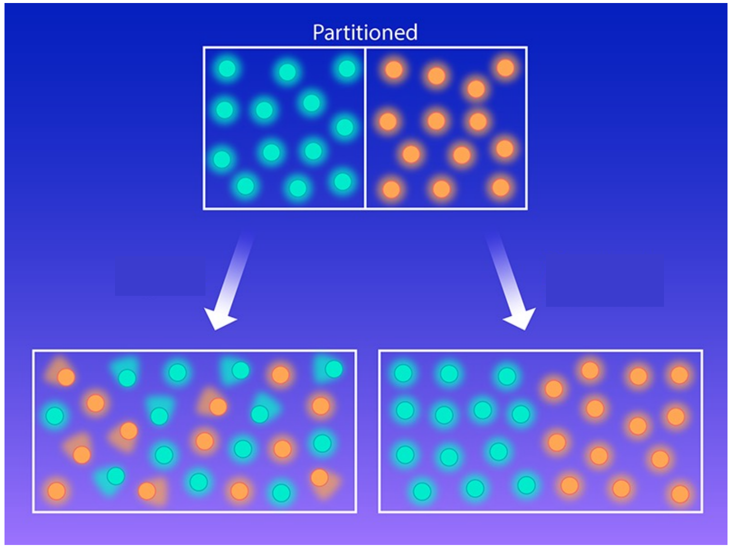
To thermalize or not thermalize, that is the question

how equilibration dictates the landscape
of out-of-equilibrium phenomena

Francisco Machado (QuTech)



Everything You Always Wanted to Know About Many-Body Dynamics* -- Feb 3

Goals for today:

- Build a language and intuition for what thermalization is in isolated quantum systems and how it can be studied
- Connect different ideas to stimulate discussions

Not Goals for today:

- Proving statements – some statements will be ~hand-wavy~
- Try to build quantum chaos from classical chaos

Today's plan:

1) What is thermalization?

2) Thermalization as a dynamical process in isolated quantum systems

3) Connections between thermalization and random matrix theory

4) Thermalization Landscape:

   Different flavors = different non-equilibrium phases

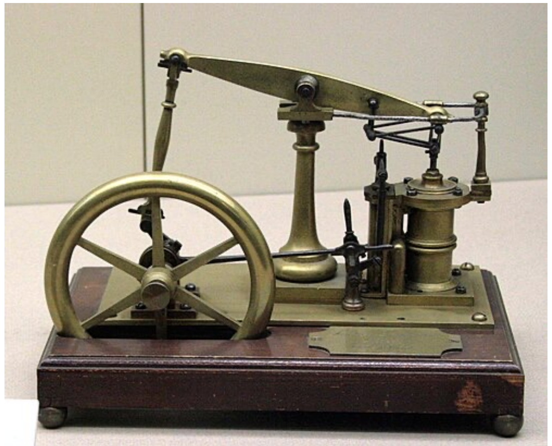
   MBL Time Crystal and Prethermal Time Crystals

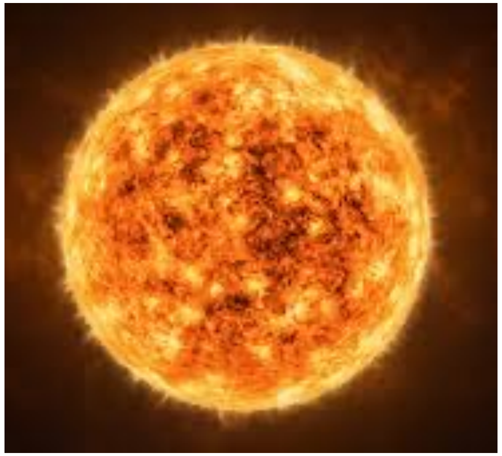
What is thermalization?

The process under which a system approaches an equilibrium steady state

What is an equilibrium steady state?

A state that remains unchanged by the dynamics of the system
characterized by a few macroscopic quantities



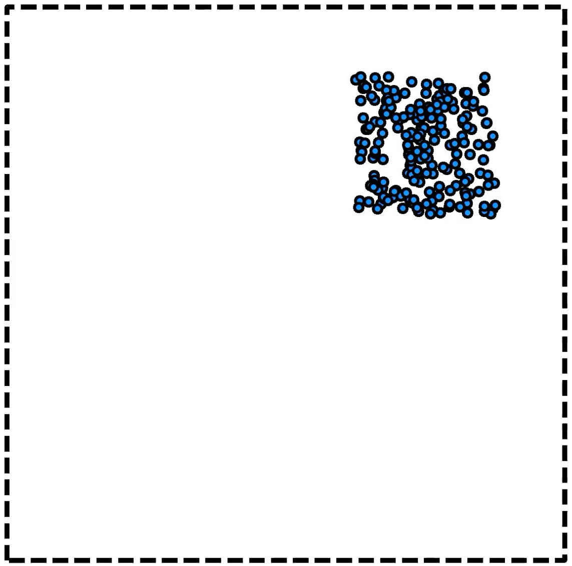


But under this definition, do systems thermalize?

But under this definition, do systems thermalize?

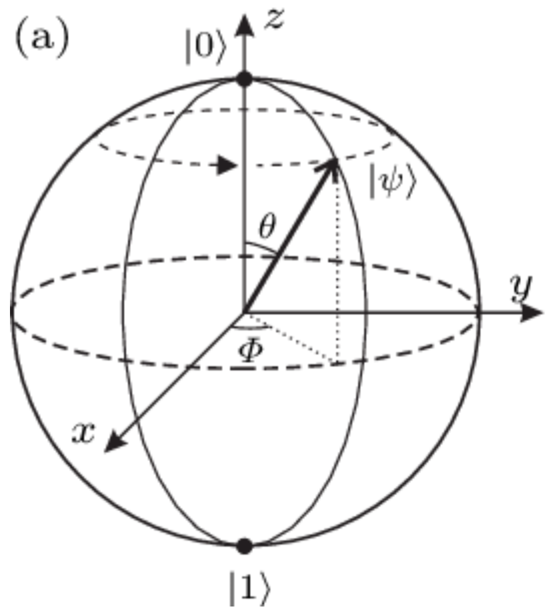
Classical System

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

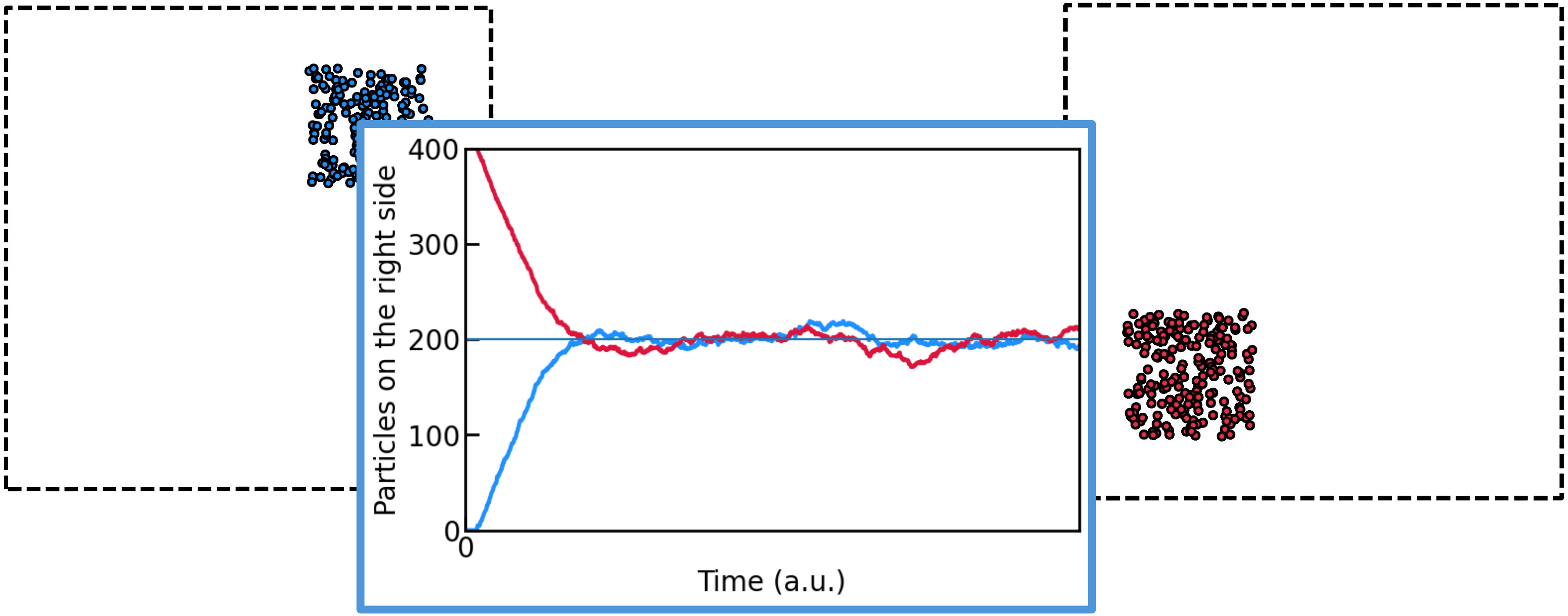


Quantum System

$$i\partial_t \left|\psi(t)\right\rangle = H \left|\psi(t)\right\rangle$$

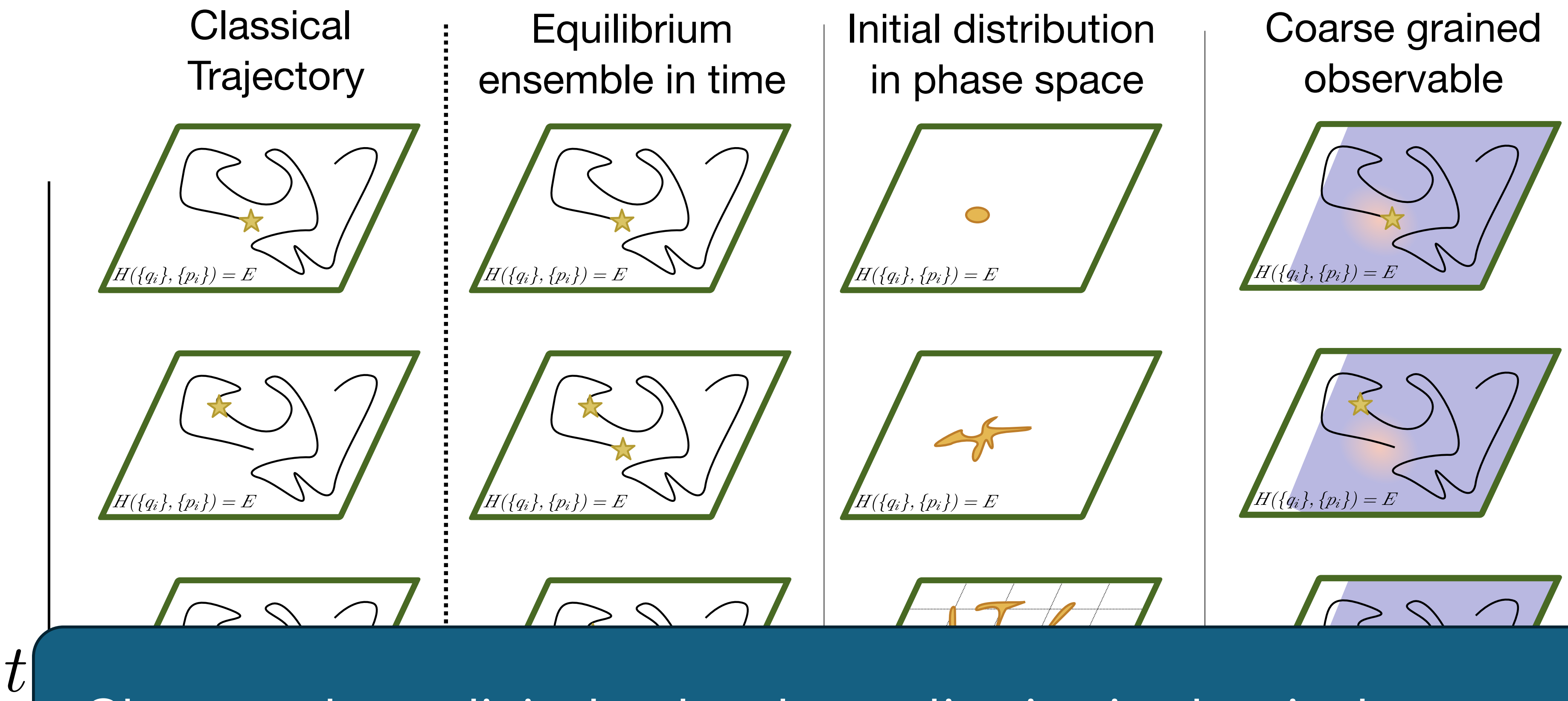


Yes, but need to focus on the right thing!



Global properties behave differently

Need some form of averaging

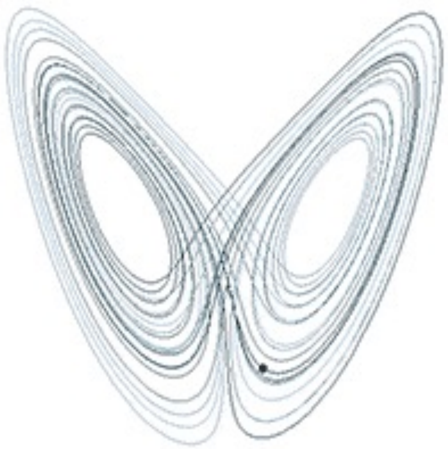


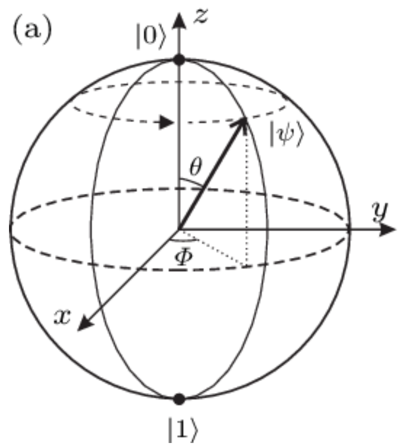
Chaos and ergodicity lead to thermalization in classical systems

Quantum changes the rules of the game

Unfortunately: Classical trajectory != quantum state dynamics

Non-linear dynamics



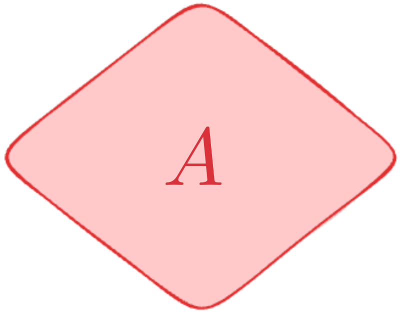


Simple linear evolution (no chaos)

Fortunately: Entanglement = new approach to "average"

Seeing the equilibrium state in a different way

$$|\psi(t)\rangle = \sum_i \lambda_i |\phi_i^A\rangle \otimes |\phi_i^{\bar{A}}\rangle$$

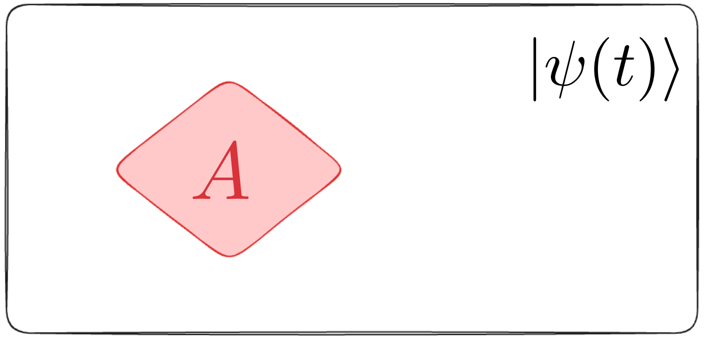


$$\rho_A = \mathrm{tr}_{\bar{A}}\left[\,|\psi(t)\rangle\langle\psi(t)|\,\right] = \sum_i |\lambda_i|^2 |\phi_i^A\rangle\langle\phi_i^A|$$

Thermalization: $\rho_A \overset{?}{\to} \mathrm{tr}_{\bar{A}}\left[\frac{e^{-\beta H}}{Z}\right]$

$\forall O_A \in A \quad \langle\psi(t)|\, O_A(t)\, |\psi(t)\rangle \to \mathrm{tr}\left[O_A \frac{e^{-\beta H}}{Z}\right]$

Thermalization as a dynamical process (in isolated quantum systems)



Initial state: $|\psi_0\rangle = \sum_n c_n |n\rangle$ $H|n\rangle = E_n |n\rangle$

Evolving under: $i\partial_t |\psi(t)\rangle = H|\psi(t)\rangle$

$$\langle\psi(t)| O_A |\psi(t)\rangle = \sum_{n,m} c_n^* c_m \langle n| O_A |m\rangle e^{i(E_n - E_m)t}$$

$$= \sum_n |c_n|^2 O_{nn} + \sum_{n \neq m} c_n^* c_m e^{i(E_n - E_m)t} O_{nm}$$

Thermalization of local observables

$$\langle\psi(t)|\, O_A \,|\psi(t)\rangle = \sum_n |c_n|^2 O_{nn} + \sum_{n\neq m} c_n^* c_m e^{i(E_n - E_m)t} O_{nm}$$

Constant term that depends on overlaps with eigenstates and diagonal matrix elements

Summ of exponentially many oscillating terms each with its own frequency

$$\langle\psi(t)|\, O_A \,|\psi(t)\rangle \to \sum_n |c_n|^2 O_{nn}$$

Thermalization of local observables

$$\langle\psi(t)|\, O_A\, |\psi(t)\rangle \rightarrow \sum_n |c_n|^2 O_{nn}$$

Necessary ingredients:

- Fluctuations must be small

  Large number of terms (thermodynamic limit)

  Frequencies are different

- Constant term must match thermal value

These features are encoded in the **Eigenstate Thermalization Hypothesis**

Eigenstate Thermalization Hypothesis (ETH)

Review: D'Alessio et al *Advances in Physics (2016)*

Posits that:
thermalization occurs because each eigenstate is a good micro-canonical ensemble

$$O_{mn} = O(\bar{E})\delta_{mn} + e^{-S(\bar{E})/2} f_O(\bar{E}, \omega) R_{mn}$$

Equilibrium values

Fluctuations are suppressed by the entropy S

Smooth function

Random variables

$$R_{mn} = \mathcal{N}(0, 1)$$

$$\bar{E} = (E_n + E_m)/2$$

$$\omega = E_n - E_m$$

ETH as a refinement of Random Matrix Theory

What are the properties of observables if the Hamiltonian is a purely random matrix?

$$O_{mn}^{(RMT)} = \overline{O}\delta_{mn} + \sqrt{\frac{\overline{O^2}}{\mathcal{D}}}R_{mn}$$

$\mathcal{D}$ **- Hilbert Space size**

1) Every eigenstate is equivalent
2) Diagonal term encodes the trace of the operator
3) Fluctuations in off-diagonal terms are suppressed by $\mathcal{D}$

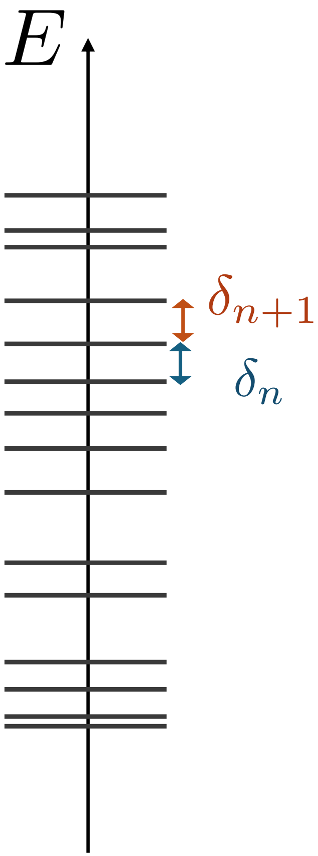
In ETH, the average and fluctuation size depend on the energy of the eigenstate

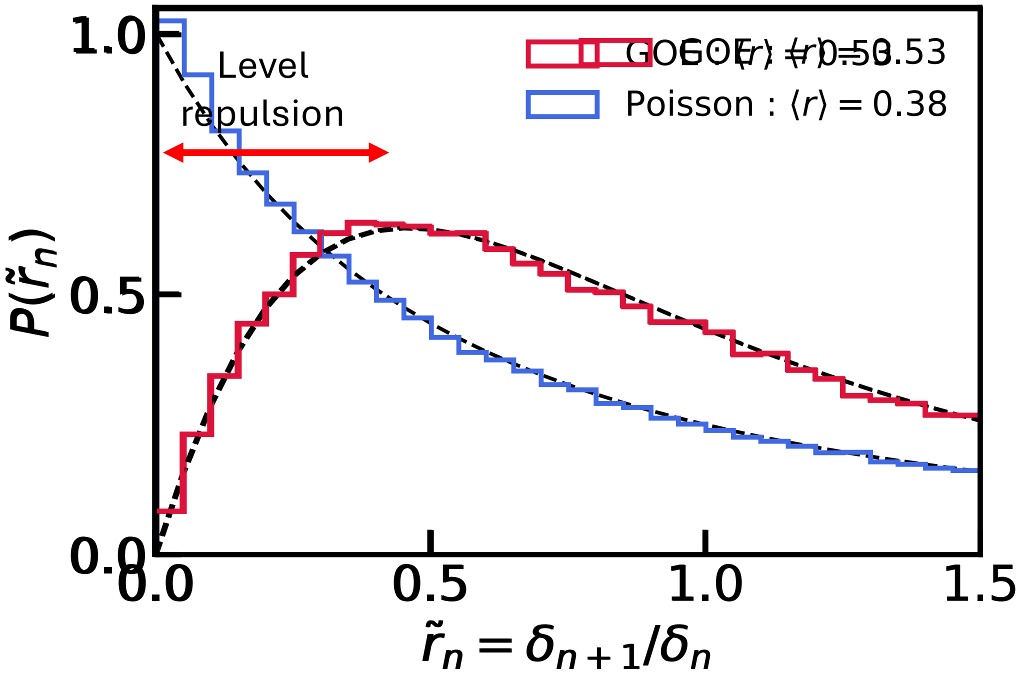
$$O_{mn} = O(\bar{E})\delta_{mn} + e^{-S(\bar{E})/2} f_O(\bar{E}, \omega) R_{mn}$$

GOE: $H_{ij} = \mathcal{N}(0,1)$

Poisson:
independent modes

RMT, level statistics and ergodicity





RMT, level statistics and ergodicity -- GOE

Easier to understand in terms of perturbing the spectrum

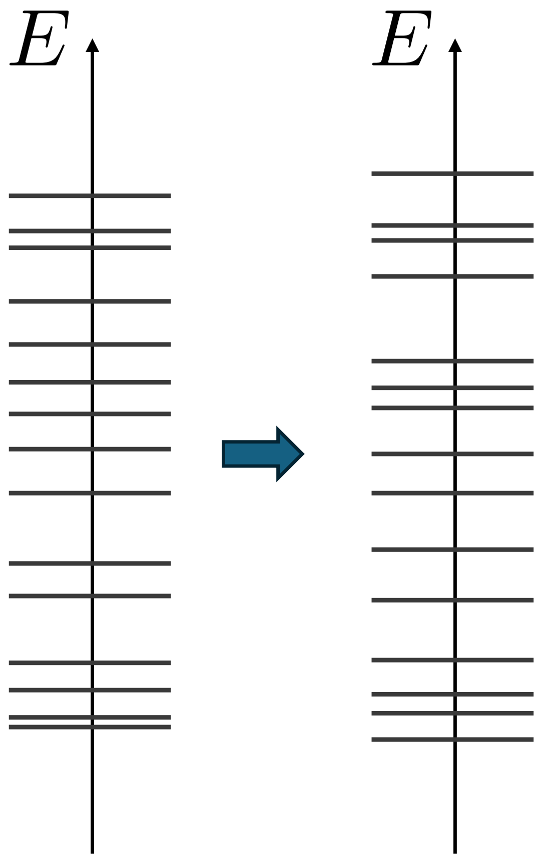
$$H = H_0 + \epsilon\Lambda$$

$$\delta E_n = \epsilon \langle n| \Lambda |n\rangle + \epsilon^2 \sum_m \frac{\langle n| \Lambda |m\rangle \langle m| \Lambda |n\rangle}{\delta_{nm}}$$

If $|n\rangle$ are delocalized in Hilbert space → $\langle n| \Lambda |m\rangle$ is non-zero

ergodic

Nearby states push each other apart and gaps do not close!



RMT, level statistics and ergodicity -- Poisson

Easier to understand in terms of perturbing the spectrum

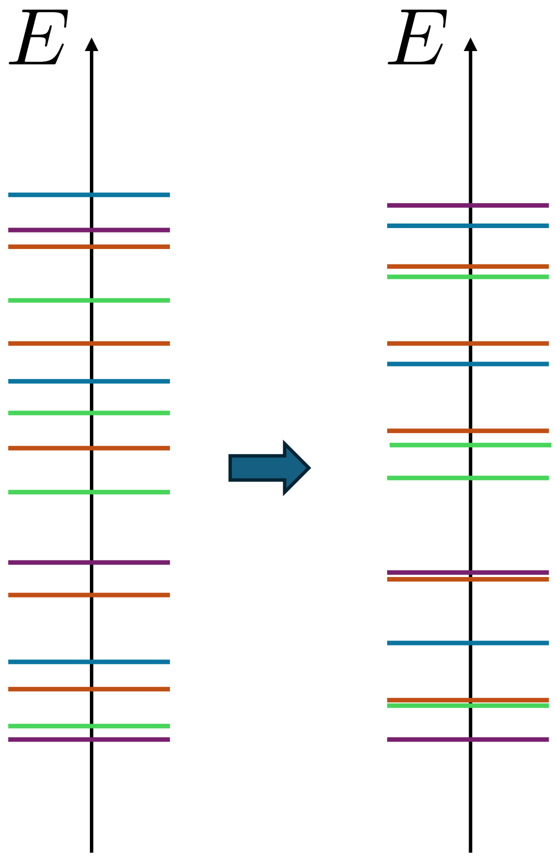
$$H = H_0 + \epsilon\Lambda$$

Symmetry, particle number, etc...

Different states live in different "sectors"; unless Λ connects different sectors, the off-diagonal matrix element is, in general, zero

$$\langle n| \Lambda |m\rangle = 0$$

Locally, state's energy is independent: Poisson statistics!



Quick Summary

- In quantum systems, thermalization is naturally defined in terms of subsystems, and as a dynamical process of observables relaxing

- A Hamiltonian whose statistics match that of a random matrix thermalizes
  - Computing these statistics for specific instances help us diagnose the presence of structure that prevent thermalization

[Quick Aside 1] Proving thermalization

Can you prove thermalization?

Not in general: it is an undecideable problem

For the curious:
Can simulate a Turing machine in the dynamics of a specific Hamiltonian and initial state, maping the question of thermalizing into a halting problem

However, by looking at typical states, in certain classes of Hamiltonians one can make headway!

Examples:
- *Quantum mechanical evolution towards thermal equilibrium
  Linden et al PRE (2009)
- Quantum thermalization must occur in translation-invariant systems at high-temperature
  Pilatowsky-Cameo, Choi in Nature Comm (2025)

Common assumption: "We emphasize that the restriction to Hamiltonians that have no degenerate energy gaps is an extremely natural and weak restriction."*

Structured eigenspectra exhibit stronger and faster recurances that prevent thermalization

[Quick Aside 2] Where did the information go?

Store 1-bit of information:

$$O(t=0)\left|\psi_0\right\rangle = \pm\left|\psi_0\right\rangle$$

Is still present at late time

$$O(-t)\left|\psi(t)\right\rangle = \pm\left|\psi_0(t)\right\rangle$$

<u>What is the operator I need to measure to recover the information?</u>

Heisenberg evolution of the operator:

$$\frac{d}{dt}O(t) = i[H, O(t)]$$

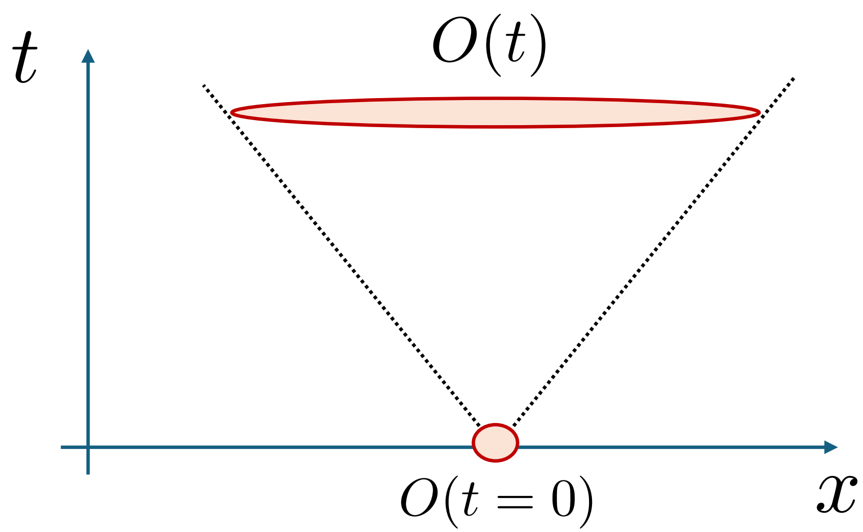
$$H = \sum_i \lambda_i \hat{h}_i \otimes \hat{h}_{i+1} \qquad O(t=0) = \sigma_0^x$$

$$\frac{d}{dt}O(t) = \sum_\alpha a_\alpha \sigma_0^\alpha + \sum_{\alpha,\beta}(b_{\alpha\beta}\sigma_0^\alpha\sigma_1^\beta + c_{\alpha\beta}\sigma_0^\alpha\sigma_{-1}^\beta)$$

Review: Xu and Swingle PRX Quantum (2024)

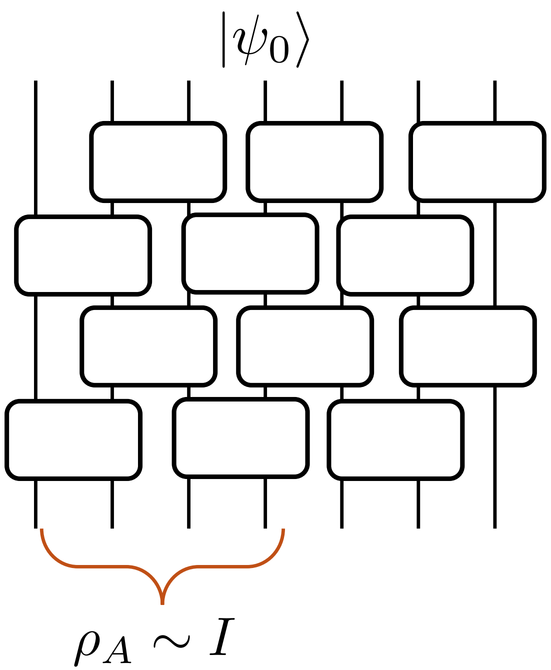
Operator grows more complex, and more extended in space

This is known as <u>scrambling</u>

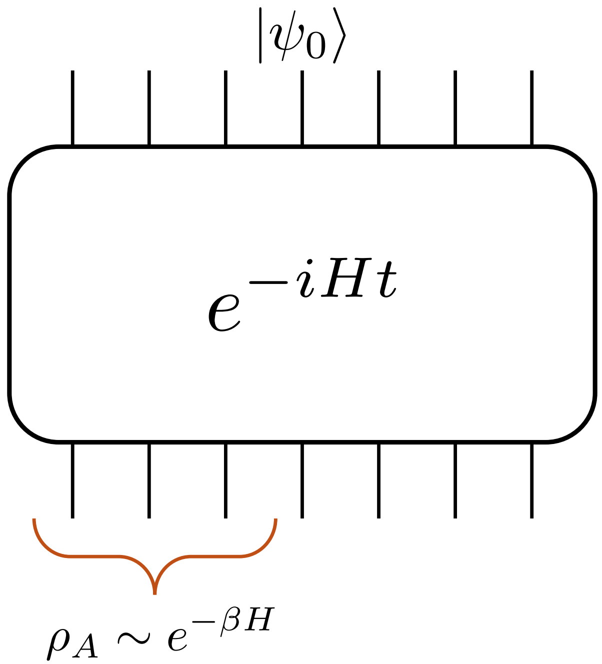


Beyond Hamiltonian dynamics

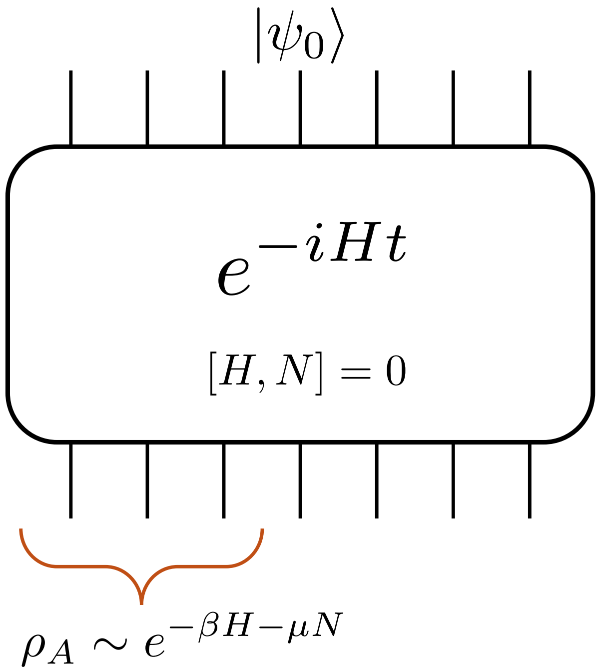
Random Unitary Circuit



Hamiltonian evolution



Hamiltonian evolution w/ symmetries



Energy acts as a constraint!

Landscape of Thermalizing Dynamics

Fully Thermalizing — **Fully Non-Thermalizing**

Periodically driven system

$$H(t) = H(t + \tau)$$

Many-Body Localization

Hamiltonian evolution

Hilbert space fragmentation

Random Unitary
Circuit

Hamiltonian evolution w/
symmetries or constraints

Integrable systems

Floquet
prethermalization

Many-body scars

Different thermalization regimes yield different kinds of out-of-equilibrium phases

Parting words

- Thermalization as a complex yet generic feature in quantum systems

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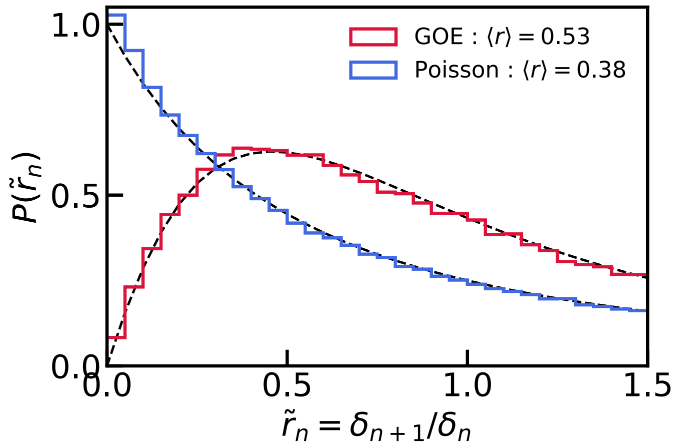
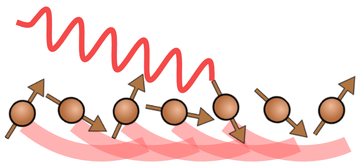
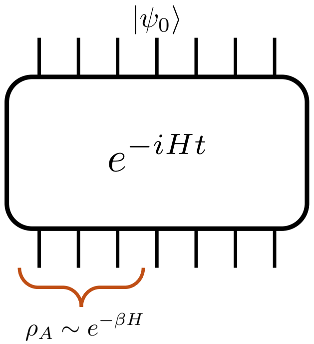
Admits a statistical framework

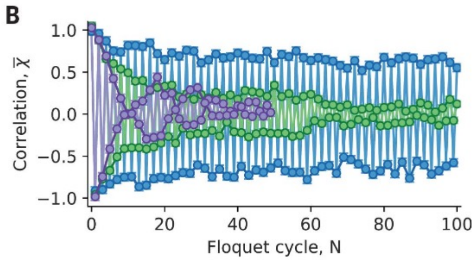






- Thermalization comes in many shapes and sizes: not yet have a full understanding of the landscape

- Different thermalization dynamics enable different out-of-equilibrium phenomena



Thank you!