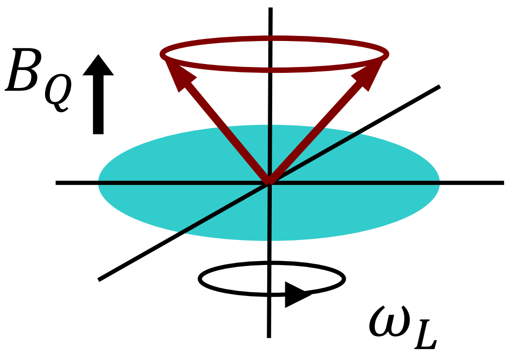
Magnetic resonance experiments:
Standard experimental settings and theoretical concepts

Almost all modern approaches to dynamics and control of spins and qubits are based on magnetic resonance-type settings.

Single spin or ensemble of spins in a strong magnetic field B_Q



"Quantizing field" – determines primary quantization axis

(why "primary"? Will see shortly)

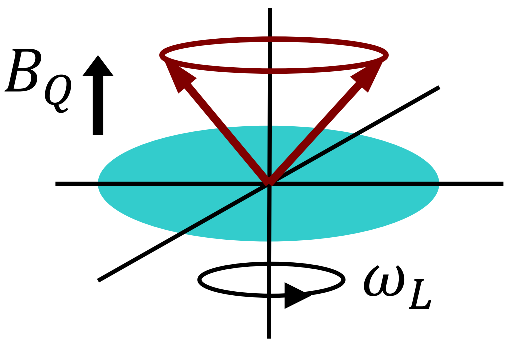
Induces Larmor precession with the frequency ω_0

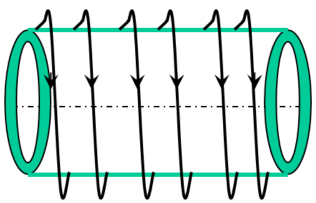
$$H = \omega_L S_z + H'$$

Informally we say that $H' \ll \omega_L$, i.e. typical energies of H' are much smaller than ω_L. But it is H' that is usually of most interest.

How do we excite ("drive") spin dynamics?

Pointer states $|\uparrow\rangle$ and $|\downarrow\rangle$. How do we make spins move?
Oscillating field h along X, frequency ω_L – performs rotations.
Drives spins out of equilibrium state. How?

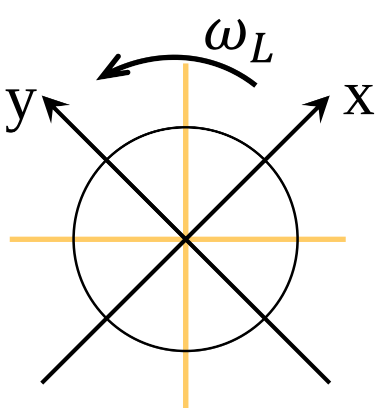




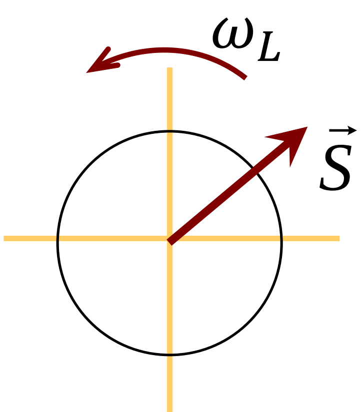
$$h_x(t) = 2h\cos\omega_L t$$

Key concept: **Rotating frame**

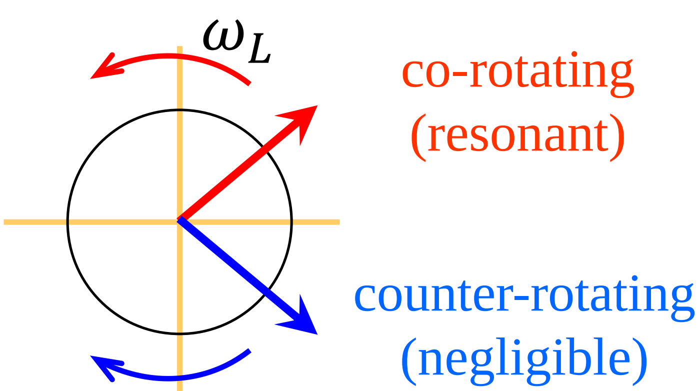
Rotating frame



Spin



Oscillating field



Rabi oscillations and rotating frame

We are interested only in dynamics at frequencies close to ω_L
Driving at the frequency ω close to ω_L : $|\omega - \omega_L| \ll \omega_L$, $h \ll \omega_L$

$$H = \omega_L S_z + h_x(t) S_x = \omega_L S_z + 2h S_x \cos(\omega t + \alpha)$$

Rotating frame transformation: $W = \exp(-i\omega t S_z)$
Rotation with frequency ω (<u>not</u> ω_L!) around the Z-axis

$U(t) = W \cdot U_R(t)$, $U_R(t)$ – ev.op. in the rotating frame

$$i \dot{U} = H U(t) \quad \Rightarrow \quad i \dot{U}_R = \left(W^\dagger H W - \omega S_z\right) U_R =$$

$$= \left[(\omega_L - \omega) S_z + 2h \cos(\omega t + \alpha)\ \mathrm{e}^{i\omega t S_z} S_x \mathrm{e}^{-i\omega t S_z}\right] U_R =$$

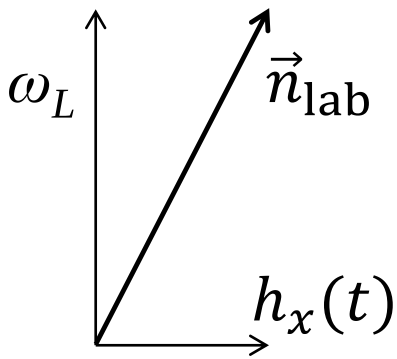
$$= \left[(\omega_L - \omega) S_z + h\left(S_x \cos\alpha + S_y \sin\alpha\right) + \{\text{nonsec. terms}\}\right] U_R$$

<u>Secular</u> Hamiltonian : $H_R = (\omega_L - \omega) S_z + h\left(S_x \cos\alpha + S_y \sin\alpha\right)$

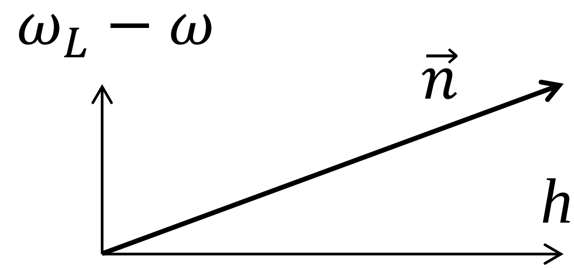
Rabi oscillations and rotating frame

Weak ($h \ll \omega_L$) time-dependent driving became strong ($h \sim |\omega_L - \omega|$) and time-independent

Lab frame:



Rotating frame:



Driven spins rotate around $\vec{n}$, S_z oscillates: **Rabi oscillations**.

Isidor Rabi [Rah-bee], Nobel prize 1944

Rotating frame: we transform $H \to H_R$, density matrix $\rho \to \rho_R$, …

But not the observables!

Different from interaction representation, rotating-wave approx., etc.

1) Inconvenient: time dependence of $S_{x,y}$, mutual dependence,…

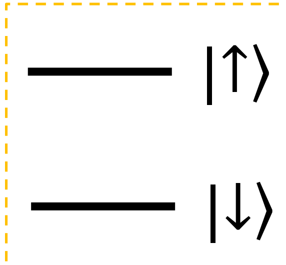
2) Rotating-frame observables are what is actually detected in standard resonance experiments (I/Q channels)

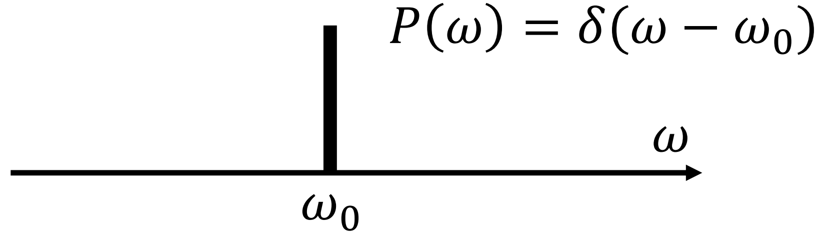
See e.g. C. P. Slichter, "Principles of Magnetic Resonance"

Spin driving: qualitative picture

Ideal:

$\hat{H} = \omega_0 \hat{S}_z$

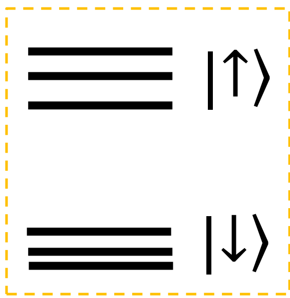


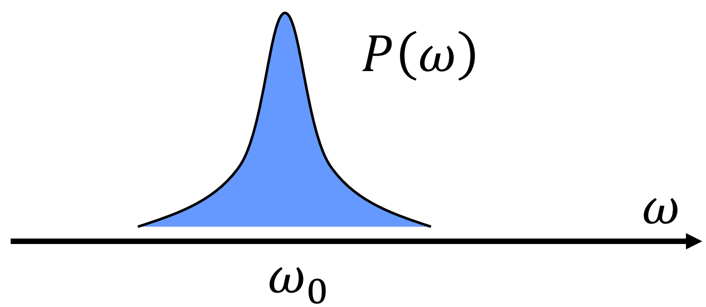


Single sharp line

Noise:

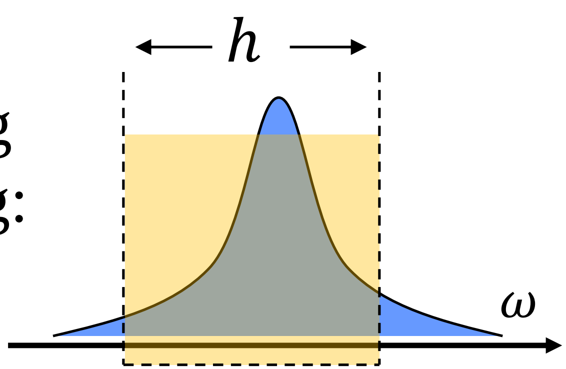
$\hat{H} = \omega\, \hat{S}_z$





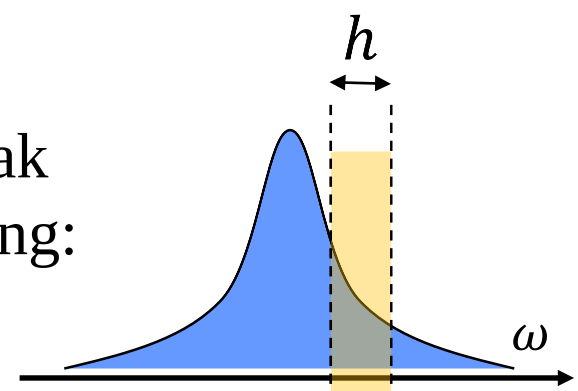
Broadened line, determined by $P(\beta)$

Strong driving:



Spins resonant with driving

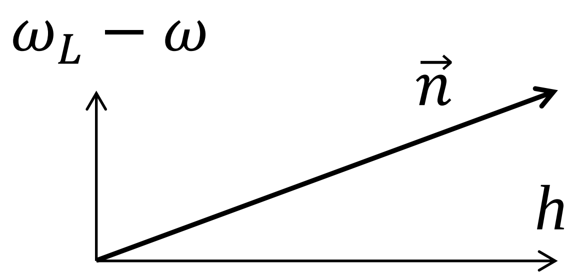
Weak driving:



Spins resonant with driving

Rabi oscillations: strong driving

Rotating frame:



$$H = B\, S_z + h\, S_x \quad \text{with} \quad B = \omega_L - \omega$$

How do we take noise into account ?

Consider quasi-static noise: **inhomogeneous broadening**

Weak non-resonant fields (ω', B') : $|\omega_0 - \omega'| \gg B'$
If directed along X- and Y-axes: non-secular, neglect

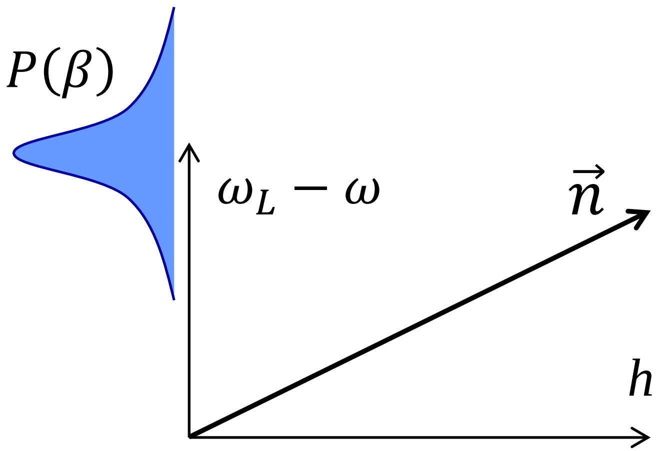
Therefore, only static noise along Z-axis matters: $B = B_0 + \beta$

$$\omega_L = \omega_0 + \beta \ , \ \text{e.g.} \ \ P(\beta) = \frac{1}{\sqrt{2\pi b^2}} \exp\left(-\frac{\beta^2}{2b^2}\right)$$

This is precisely what we studied in the previous lecture

Rabi oscillations: strong driving

Rotating frame:



$$H = B\,S_z + h\,S_x\,,$$

$$B = \omega_L - \omega$$

<u>Strong off-resonant driving:</u> $h, B_0 \gg b$

Initial state decays towards equilibrium (pointer) states
Pointer states – already analyzed earlier in detail

Decay of all components has Gaussian form, decay time T_2^*:

$$T_2^* = b^{-1} \qquad \text{for } B_0 \gg h$$

$$T_2^* = b^{-1} \cdot (h/B_0) \quad \text{for } B_0 \ll h$$

Any questions? <u>Hint</u>: $T_2^* = b^{-1}(h/B_0)$ for $B_0 \ll h$. What about $B_0 = 0$?

Rabi oscillations, strong driving

Strong resonant driving: $B_0 = 0\,,\; h \gg b$

Resonant: $\omega = \omega_0$, i.e. exactly at resonance with the line center
Spectroscopic language: **driving saturates the line**

First order in β is gone, but there is second:

$$\Omega = \Omega_0 + B_0 \cdot (\beta/\Omega_0) \; + \; [h^2/2\Omega_0] \cdot (\beta/\Omega_0)^2 + \cdots$$

For $B_0 = 0$:
$$\Omega = \sqrt{B^2 + h^2} \equiv \sqrt{\beta^2 + h^2} \approx h + \beta^2/(2h)$$
$$\varphi \approx ht + t\,\beta^2/(2h)$$

The integral which determines the form of decay:

$$\int P(\beta) d\beta \, \mathrm{e}^{it\beta^2/(2h)} = \frac{1}{\sqrt{2\pi b^2}} \int \mathrm{e}^{i\,\frac{t\beta^2}{2h}} \, \mathrm{e}^{-\frac{\beta^2}{2b^2}} \, d\beta = \left(1 - i\,\frac{b^2 t}{2h}\right)^{-1/2} =$$

$$= r(t) \cdot \mathrm{e}^{i\mu(t)} \,, \quad \text{with } \; r = \left[1 + \left(\frac{b^2 t}{2h}\right)^2\right]^{-1/4} \; \text{ and } \; \mu = \frac{1}{2} \tan^{-1} \frac{b^2 t}{2h}$$

Rabi oscillations, strong resonant driving

Leading order in $b/h \ll 1$: decaying rotation around X-axis

Explicit results:

$$\hat{S}_x(t) = \hat{S}_x \text{ (constant in time),}$$

$$\hat{S}_z(t) = r(t)\left[\hat{S}_z \cos\vartheta + \hat{S}_y \sin\vartheta\right]$$

$$\hat{S}_y(t) = r(t)\left[\hat{S}_y \cos\vartheta - \hat{S}_z \sin\vartheta\right]$$

where $\vartheta = ht + \mu(t)$

All operators are understood as averaged over the noise, the symbol $\langle \ldots \rangle_\beta$ omitted

Initial decay at $t \ll h/b^2$:
quadratic, $r(t) \approx 1 - At^2$, with linearly changing phase $\mu(t)$, i.e.
Rabi frequency is renormalized by $b^2/(2h)$.
Looks like regular Gaussian, but…

Long-time decay at $t \gg h/b^2$:
extremely slow power-law decay, $r(t) \propto t^{-1/2}$ with $\mu \approx \pi/4$

Rabi oscillations, strong resonant driving

Leading order in $b/h \ll 1$: quantization axis $\vec{n}$ along X-axis

Note 1: this is in the rotating frame! In the lab frame the quantization axis is precessing around the Z-axis, and the pointer states have explicit time dependence.

Recall the comments from the previous lecture: pointer states can depend on time.

Note 2: we have quasi-equilibriuim (pointer) states along the rotating-frame X axis: $\frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle)$

Same idea is used in quantum optics: the concept of "dressed states".
Atomic states change under strong optical driving, become an equal-weight superposition of $|g\rangle$ and $|e\rangle$.
But the theory is more complex: quantum photons instead of classical driving field, account of spontaneous decay, etc.

Rabi oscillations, strong driving: comments

- Strong driving changes the oscillation frequency of $S_{x,y}$:

from B_0 to $\Omega_0 = \sqrt{B_0^2 + h^2} \approx B_0 + h^2/(2B_0)$

Similar phenomenon in quantum optics: ac Stark shift of atomic levels under strong optical driving

- The power-law decay $t^{-1/2}$ has no well-defined lifetime

(formally, Rabi rotations in this regime have infinitely long lifetime)

Jumping ahead: cw spectroscopy would show two lines, split by h

These lines are narrow: Fourier transform of $t^{-1/2}$ is $\omega^{-1/2}$

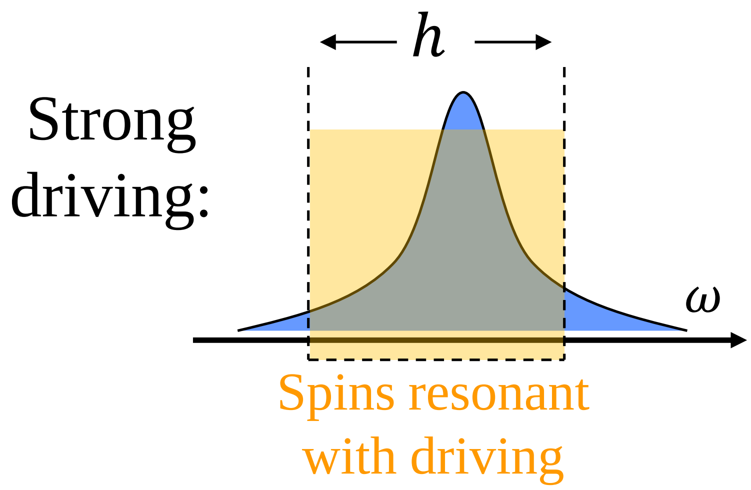
(formally, infinitely narrow)

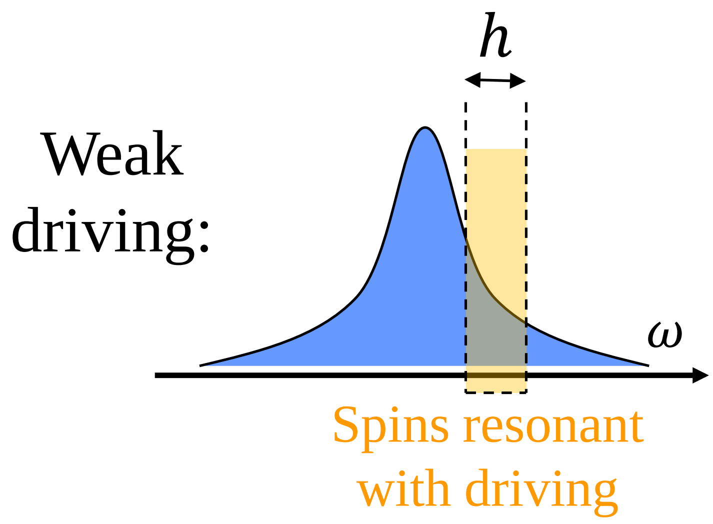
Similar phenomenon in quantum optics: Autler-Townes splitting

- Transition from resonant to non-resonant regime occurs when first and second orders are similar: $B_0 \cdot (\beta/\Omega_0) \sim [h^2/2\Omega_0] \cdot (\beta/\Omega_0)^2$

In this case B_0 is small, and the transition occurs when $B_0 \sim b$

Rabi oscillations: weak driving





Strong driving, weak noise ($h \gg b$) : **driving saturates the line**

Weak driving: the regime of continuous-wave (**cw**) **spectroscopy**

Typical initial state: along Z-axis, with $\langle S_x(0) \rangle = \langle S_y(0) \rangle = 0$

Driving at frequency ω : can use previous results

$$\langle S_z \rangle(t) = \langle S_z^0 \rangle \cdot [1 - 2\, n_x^2 \sin^2(\varphi/2)]$$

$$\varphi = \Omega t \ , \quad \Omega = \sqrt{h^2 + B^2} \ , \quad n_x = h/\Omega$$

Rabi oscillations: weak driving

Consider general line shape $P(\beta)$ with characteristic width b

$$\langle S_z(t)\rangle = \langle S_z(0)\rangle \cdot \left[1 - h^2 \int P(\beta) d\beta \cdot 2\,\frac{\sin^2\left(\frac{t}{2}\sqrt{h^2+\beta^2}\right)}{h^2+\beta^2}\right]$$

We focus on “perturbative long” times: $ht \ll 1$, but $bt \gg 1$

$P(\beta)$ is assumed to vary smoothly with β, on a scale $\beta \sim b$

In contrast, $2\,\frac{\sin^2\left(\frac{t}{2}\sqrt{h^2+\beta^2}\right)}{(h^2+\beta^2)}$ at large t (but small ht!) has a sharp peak of small width ($\propto t^{-1}$) and large height ($\propto t$) at $\beta = 0$; decays fast as $|\beta|$ grows.

If you have a déjà vu feeling – you are right.
This is just standard time-dependent perturbation theory,
derivation of the **Fermi Golden Rule**

If you want more mathematical clarity (optional).

1) Set $ht = 0$ for convenience: this just means that we are working in the leading order w.r.t. the small parameter ht. We should study $I = \int P(\beta)\, \frac{\sin^2(\beta t/2)}{\beta^2}\, d\beta$.

2) $P(\beta)$ has a typical scale b: means that it depends on dimensionless quantity $x = \beta/b$, and there are no other parameters in this function, large or small.

So we re-write $I = \int P(\beta)\, \frac{\sin^2(\beta t/2)}{\beta^2}\, d\beta = \int p(x)\, \frac{\sin^2(x{\cdot}bt/2)}{(x{\cdot}b)^2}\, dx$

<u>Important</u>**:** $P(\beta)$ is dimensional quantity, with dimensionality $[\mathrm{rad/s}]^{-1}$

Because $\int P(\beta)\, d\beta = 1$, and $[d\beta] = [\beta] = \frac{\mathrm{rad}}{\mathrm{s}}$

Thus, $p(x) = b \cdot P(bx)$ and $p(x)\, dx = P(\beta)\, d\beta$

3) Use large dimensionless parameter $\theta = bt$, consider $\theta \to \infty$.

$I = \int p(x)\, \frac{\sin^2(x{\cdot}bt/2)}{(x{\cdot}b)^2}\, dx = \left(\frac{t}{b}\right) \cdot \int p(x)\, \frac{\sin^2(\theta{\cdot}x/2)}{\theta{\cdot}x^2}\, dx$, and (t/b) is a parameter

(it is dimensional, so it is neither small nor large – just a quantity)

4) Can show: $g(x) = 2\, \frac{\sin^2(\theta{\cdot}x/2)}{\theta{\cdot}x^2} = \frac{1-\cos\theta x}{\theta x^2} \;\to\; \pi \cdot \delta(x)$ when $\theta \to \infty$

Consider the integral I at $\theta \to \infty$, extend it to complex plane, calculate the residue at $x = 0$.

Get the answer $I = (\pi\, t/b) \cdot p(0) = \pi t \cdot P(0)$

<u>**Note**</u>: condition $bt \gg 1$ is the key here; $ht \ll 1$ is not crucial, can set $ht = 0$

Weak driving and cw spectroscopy

So we obtain: $\langle S_z(t)\rangle = \langle S_z(0)\rangle \cdot [1 - \pi\, h^2 t\, P(0)]$

$\frac{d\langle S_z\rangle}{dt} = -\pi h^2 P(0)$ or, in the lab frame: $\frac{d\langle S_z\rangle}{dt} \propto P(\omega)$

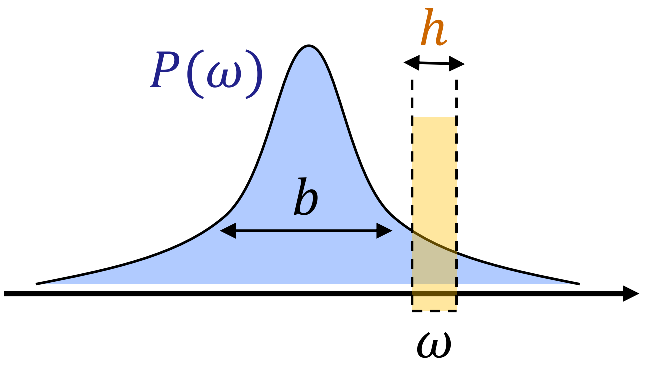
Physical meaning? Initially, the spin is along quantizing field
Start driving it, $\langle S_z\rangle$ decreases: spin flipped, energy absorbed.
Absorbed power $W \propto d\langle S_z\rangle/dt$

Measuring absorption while continuously driving spins at frequency ω directly measures the line shape $P(\omega)$

Continuous-wave (cw) spectroscopy

Most basic characterization of noise and relaxation.

NMR, ESR, optics, IR,… – idea is the same



This is for $t \ll 1/h$. What happens later, when $ht \sim 1$?

Weak driving and cw spectroscopy

Serious problems with the current treatment

- Energy absorbed – from where? From the cw driving field

$$\vec{H}(t) = \vec{e}_x \cdot 2h\cos(\omega t + \alpha)$$

Absorbed power, average per oscillation period: $W = -\overline{\vec{M} \cdot d\vec{H}/dt}$

I.e., there must be non-zero components $S_x(t)$ and $S_y(t)$

More detailed theory is needed, will discuss later (Bloch-Redfield)

- Our system : ensemble of non-interacting energy-conserving spins

Initially absorb energy as described, but this cannot go on forever

Equlibrium state is not well defined. Spins keep precessing, unhindered perpetual rotation: energy flows back and forth.

There must be some process that "resets" $\langle S_z(t) \rangle$

It could be slow fluctuations of β, interaction between spins, coupling to environment, … The same problem as in standard statistical physics: origins of stochasticity, thermalization, etc.

Phenomenological Bloch equations

Phenomenological approach: there must be some relaxation process that continually resets the spins and steers them to equilibrium.

Postulate that such a process is linear and memoryless, so the system's relaxation towards equilibrium (without driving) :

$$\frac{dM_z}{dt} = -\frac{1}{T_1}(M_z - M_z^0) \qquad \frac{dM_{x,y}}{dt} = -\frac{1}{T_2}M_{x,y}$$

Relaxation time T_1 – rate of longitudinal relaxation

Relaxation of the energy of spins in the quantizing field along Z. How fast is the energy brought to or taken away from spins.

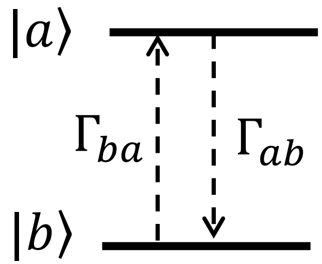
M_z^0 – equilibrium magnetization along the quantizing field

$$\frac{1}{T_2} = \frac{1}{2T_1} + \frac{1}{T_2'} \quad \text{i.e. } T_2 \leq 2T_1 \text{ – to ensure correctness}$$

Relaxation time T_2' – transverse relaxation. Does not involve energy exchange, purely internal spin relaxation (e.g. spin-spin coupling)

Phenomenological Bloch equations

Example:
two-level system
(e.g. spin 1/2)



Rate equation:

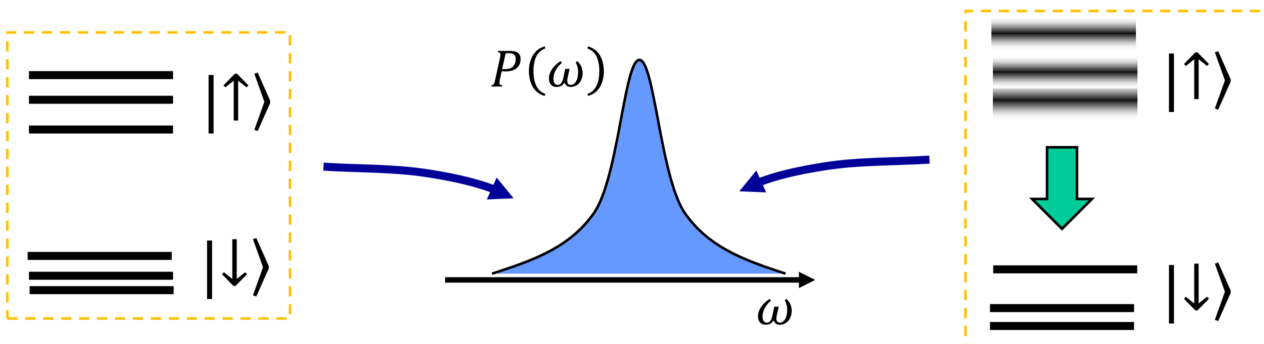
$$\frac{d}{dt}\begin{bmatrix} N_a \\ N_b \end{bmatrix} = \begin{bmatrix} -\Gamma_{ab} & \Gamma_{ab} \\ \Gamma_{ab} & \Gamma_{ab} \end{bmatrix}\begin{bmatrix} N_a \\ N_b \end{bmatrix}$$

Gives Bloch equation for M_z. But may not work for more levels!
Depends on rates, things may get complicated.

But the main idea holds: equilibrium requires equilibration mechanism

Simple spectroscopy (FT or cw) gives $P(\omega)$ but often does not say much about individual lines:

Many lines of zero width



Many lines of width $T_1{}^{-1}$

In general, response to driving depends on b, h, T_1, line shape,…
Gives rise to many useful effects (hole burning, spin echo,…)

A few notes on NMR/ESR experimental settings

Large quantizing field B_0, much larger than any other relevant energy scale. But still $g\mu_B B_0 \lll kT$, so the initial state is:

$$\rho \approx \frac{1}{Z}\exp\left(-\frac{g\mu_B B_0\, S_z}{kT}\right) \approx \frac{1}{Z}\left(\hat{1} - \epsilon S_z\right), \text{ i.e. } \rho_{\text{relevant}} \propto S_z$$

because the identity part of the density matrix is not affected much.

Dynamics in relevant experiments mostly unitary: $U\,\hat{1}\,U^\dagger = UU^\dagger = \hat{1}$
More generally: dynamics in NMR experiments is mostly unital,
i.e. maps $\hat{1} \to \hat{1}$

The signal from $\rho' = \hat{1}$ is zero: this part of density matrix contributes only to noise.

Irrelevant for the signal, important for analyzing signal-to-noise ratio.

Also, **other ways to initialize spins** are used more and more often: optical (e.g. NV centers), dynamic nuclear polarization (DNP), etc.

FT spectroscopy: qubit characterization

Often the first step in qubit characterization

0) Qubit is initialized along the Z-axis
1) Apply resonant Rabi driving for a short time, rotate spins from the Z-axis to the Y-axis (so-called $\pi/2$ pulse).

2) Measure $S_x(t)$ and $S_y(t)$: often called free induction decay (FID)

(historical reasons: design of traditional NMR experiments)

or free coherence decay, or Ramsey measurement (in modified version)

3) Analyze $f(t)$: find T_2^* , examine the form of decay, deduce which noise dephases the qubit and what are its properties.

Note: We do not have to rotate the spin all the way to Y-axis, it is enough to just provide non-zero $\langle S_x^0 \rangle$ and/or $\langle S_y^0 \rangle$.
The form of decay will be the same, only the overall amplitude of the signal will scale up or down. Will be important later.

NMR/ESR and the rotating frame: comments

- Rotating frame: potential danger from non-seqular terms. Contribution decreases slowly, as $|\omega - \omega_0|^{-2}$. If the spectral density grows faster – can accumulate and /diverge. Be careful! Relatively rare in NMR/ESR, but often happens in quantum optics.

- NMR experiments: $|\omega - \omega_0|/\omega_0 \sim 10^{-3} - 10^{-5}$, measured in ppm (part per million $= 10^{-6}$). Secular approximation in rotating frame works very well.
  But for other spins and qubits more care may be required.

- One can apply $\pi/2$ pulse along X- or Y-axis, just by choosing the phase of driving. Will be important later.

- Can use either cw spectroscopy, measuring $P(\omega)$, or FT approach, measuring $f(t)$. Usually, the two are related via Fourier transform. But depends on the system (e.g. unusually long T_1 or T_2, so that driving saturates the line)

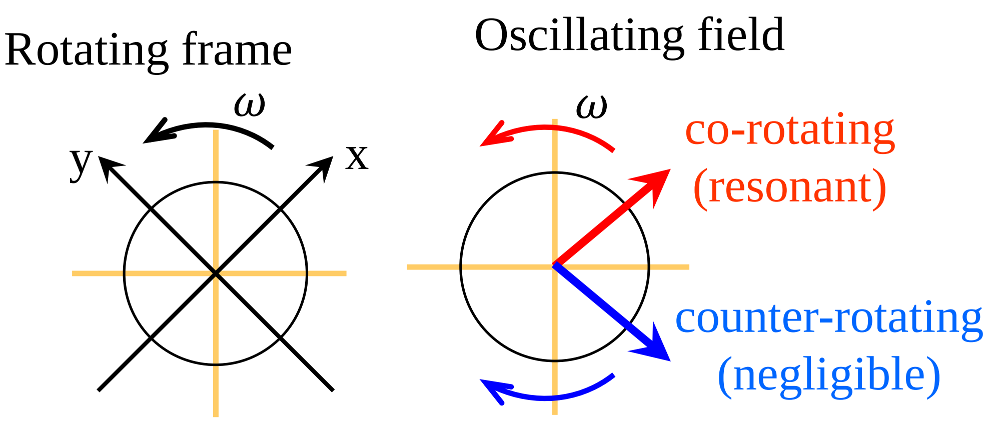
Beyond rotating-frame secular approximation

Secular approximation is of utter importance: spin motion in a general time-dependent field is not analytically solvable.

Even computers may be useless: for instance, motion of a spin under driving that is not ideally periodic (e.g. with another harmonic of incommensurate frequency) can exhibit deterministic chaos.

What lies beyond secular approximation? Two examples

1. Bloch-Siegert shift



Influence of the counter-rotating component?

Use the counter-rotating coordinate frame, neglect the co-rotating component

In this frame: quantizing field $B_Q \rightarrow$ large positive offset equal to the Larmor frequency ω_L

Compare:

Co-rotating frame: $B_Q \rightarrow$ negative offset $\omega_L \rightarrow H_z = (\omega_L - \omega) S_z$

Counter-rotating frame: $B_Q \rightarrow$ positive offset $\omega_L \rightarrow H_z' = (\omega_L + \omega)\, S_z$

I.e. in the counter-rotating frame: $H_{CR} = (\omega_L + \omega)\, S_z + h S_x$

This Hamiltonian corresponds to precession around ≈ Z-axis

(corrections to the axis are nonsecular, neglect them)

with the frequency

$$\Omega_{CR} \approx (\omega_L + \omega) + \frac{1}{2} \cdot h^2/(\omega_L + \omega) \approx (\omega_L + \omega) + h^2/(4\omega)$$

Transform back to the co-rotating frame: this rotation corresponds to a **frequency shift** $\Delta\omega = h^2/(4\omega)$

1. The shift is always positive (similar to the "effect of inverted pendulum")

2. Note the difference with ac Stark shift: Bloch-Siegert shift comes from the counter-rotating field

Beyond rotating-frame secular approximation

2. Sub-harmonic resonances

Main spin resonance: $\omega \approx \omega_L$.

Moreover, spin is a nonlinear system, so there are also resonances at

$$\omega \approx 2\omega_L\,, 3\omega_L\,, \ldots$$

But there are also sub-harmonic resonances at $\omega \approx \frac{1}{3}\omega_L\,, \frac{1}{5}\omega_L\,, \ldots$

I.e. spin is not only nonlinear, but also a parametric system

Suggested projects:

Project 1: accurate analysis of the Bloch-Siegert shift

(based on the paper of Bloch and Siegert, Phys. Rev. 57, 522 (1940), and on the textbook by A. Abragam, "Principles of Nuclear Magnetism")

Project 2: sub-harmonic resonances (based on the textbook by A. Abragam, "Principles of Nuclear Magnetism")

Brief summary:

1). Rotating frame, secular approximation. Transformation to the rotating frame is different from the rotating-wave approximation or interaction representation.

2). Rotating frame, strong driving: Rabi oscillations.
Pointer states along the total effective field, rotating in the lab frame.
Dephasing time depending on the frequency detuning $\omega_L - \omega$.

3). Strong resonant driving: extremely narrow-line pointer states along the X-axis, power-law decoherence $\propto t^{-1/2}$

4). Weak driving: cw spectroscopy, measurement of $P(\omega)$

5). FT spectroscopy: use of pulsed Rabi driving. Measuring coherence decay $f(t)$, related to $P(\omega)$ via Fourier transform.

6). Need a theory for relaxation. Bloch equations - phenomenological

7). Physics beyond rotating-frame secular theory

Bloch equations, weak driving: lineshape

$$\frac{dM_z}{dt} = -\frac{1}{T_1}(M_z - M_z^0) \qquad \frac{dM_{x,y}}{dt} = -\frac{1}{T_2}M_{x,y}$$ and assume $T_2 \sim T_1$

With field: add the Landau-Lifshits term $\dot{\vec{M}} = [\vec{M} \times \vec{H}]$

Let us use this phenomenology in the rotating frame:
driving h along the X-axis, detuning $B = (\omega_L - \omega)$ along the Z-axis

$$\frac{dM_z}{dt} = -M_y h - \frac{1}{T_1}(M_z - M_z^0) \qquad \frac{dM_x}{dt} = M_y B - \frac{1}{T_2}M_x$$

$$\frac{dM_y}{dt} = -M_x B + M_z h - \frac{1}{T_2}M_y$$

<u>Our goal</u>: consider very weak driving, $h \ll B, {T_{1,2}}^{-1}$and find the equilibrium values of the transverse components M_x and M_y

<u>Qualitative consideration</u>:
driving tilts quantization axis away from Z by small angle (of the order h/B or $hT_{1,2}$)
$M_z - M_z^0$ is of the second order in this small tilt (cosine of small angle), i.e. of the order of $(h/B)^2$ or $(hT_{1,2})^2$. If we look at linear terms in h, we can take $M_z = M_z^0$.

Bloch equations, weak driving: lineshape

$$\frac{dM_x}{dt} = M_y B - \frac{1}{T_2} M_x \qquad \frac{dM_y}{dt} = -M_x B + M_z^0 h - \frac{1}{T_2} M_y$$

$$M_+ = M_x + iM_y : \qquad \frac{dM_+}{dt} = -i\, M_+ B - \frac{1}{T_2} M_+ + i\, M_z^0 h = -Q M_+ + i\, M_z^0 h$$

$$M_+(t) = Ce^{-Qt} + \frac{i\, M_z^0 h}{Q}$$

So that in equilibrium, at $t \to \infty$, $M_+ = i\, M_z^0 h / Q$

$$M_x(\infty) = h\, M_z^0 T_2 \cdot \frac{(\omega_L - \omega)\, T_2}{1 + (\omega_L - \omega)^2\, T_2^2}$$

$$M_y(\infty) = h\, M_z^0 T_2 \cdot \frac{1}{1 + (\omega_L - \omega)^2\, T_2^2}$$

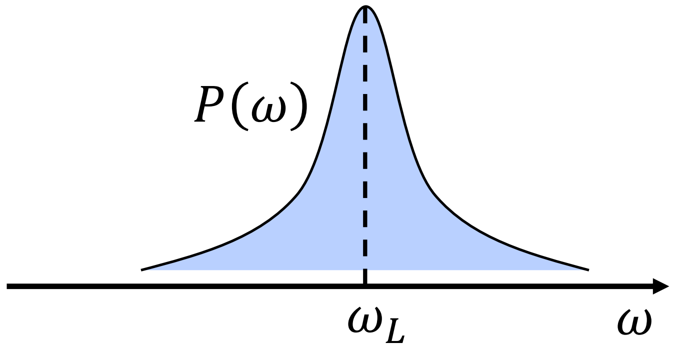
Driving in the lab frame: $\vec{H}(t) = \vec{e}_x \cdot 2h \cos \omega t$

Magnetization in the lab frame: $\vec{M}(t) = \vec{e}_x \cdot M_x \cos \omega t + \vec{e}_y \cdot M_y \cos \omega t$

Power absorbed per period: $W = -\overline{\vec{M} \cdot d\vec{H}/dt} \propto h \cdot M_y(\infty)$

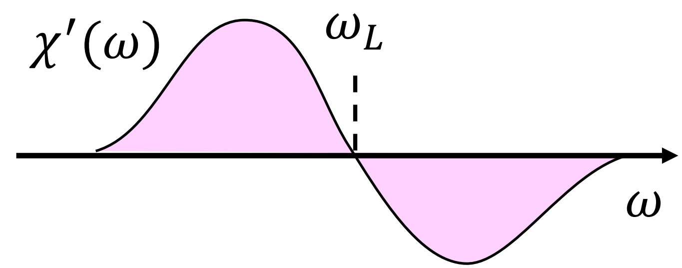
Bloch equations, weak driving: lineshape

Absorption line $P(\omega) \propto \chi''(\omega) \propto M_y(\infty)$



Lorentzian $\dfrac{1}{1+(\omega_L-\omega)^2\,T_2^2}$

Dispersion line $\chi'(\omega) \propto M_x(\infty)$



$$\frac{(\omega_L-\omega)\,T_2}{1+(\omega_L-\omega)^2\,T_2^2}$$

What is the corresponding $f(t)$?

Free decay, no driving, initial state $M_x(0) \neq 0,\ M_y(0) = M_z(0) = 0$

$$\frac{dM_x}{dt} = M_y B - \frac{1}{T_2} M_x \quad \frac{dM_y}{dt} = -M_x B - \frac{1}{T_2} M_y \quad \Rightarrow \quad M_x(t) = M_x(0)\,\mathrm{e}^{-t/T_2}\cos Bt$$

$$f(t) = \mathrm{e}^{-t/T_2}\cos(\omega_L-\omega)t \quad \xleftrightarrow{\text{Fourier Transform}} \quad P(\omega) = \frac{1}{1+(\omega_L-\omega)^2\,T_2^2}$$

Note: this is no longer a static noise field along Z, but still $f(t) \xleftrightarrow{FT} P(\omega)$