Chaotic Rabi Oscillations under Quasiperiodic Perturbation

Y. Pomeau

Groupe de Physique des Solides, Laboratoire de Physique de l'Ecole Normale Supérieure, 75005 Paris, and Service de Physique Théorique, Centre d'Etudes Nucléaires de Saclay, 91191 Gif-sur-Yvette Cedex, France

and

B. Dorizzi and B. Grammaticos

Centre National d'Etudes des Télécommunications, 92131 Issy les Moulineaux, France

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We have found quantum chaos in a system consisting of two levels coupled through a time-dependent perturbation, when this perturbation is quasiperiodic with two incommensurate frequencies. This contrasts with the case of a monochromatic perturbation where the oscillations of the physical observables are nonchaotic, because of the Floquet theorem. The existence of chaos is confirmed by the observation of rapidly decreasing autocorrelation functions, and by a continuous Fourier spectrum for time-dependent fluctuations of physical observables.

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The manifestations of quantum chaos and, in fact, the concept itself are a major problem in present-day studies of dynamical systems.[1] In the time-independent quantal case, significant progress has been made these last few years and a pattern is emerging, making possible the characterization of chaos in quantum physics.[2] Nonperturbative results on time-dependent systems, even extremely simple ones, are scarce, although some progress has been made in this direction recently.[3-5]

As far as physical applications are concerned, one often obtains a realistic description by considering systems with a few quantum levels. Thus, it seems natural to investigate the possibility of the appearance of chaos in such cases. In this Letter, we will examine a well-known and familiar system, namely a two-level system under the influence of a time-dependent perturbation. Several physical, both theoretical and experimental, realizations of such a system exist. In order to fix the ideas, we can visualize our system as modeling the multiphoton dynamics of a two-level atom illuminated by an intense light beam, or as a spin-$\frac{1}{2}$ system under the influence of a time-dependent magnetic field ($\hbar = 1$):

$$i\dot{\psi}_1 = \omega_1\psi_1 + S(t)\psi_2,$$

$$i\dot{\psi}_2 = \omega_2\psi_2 + S(t)\psi_1.$$

The latter system has been the object of several studies. In fact, the name "Rabi oscillations"[6] was coined for the description of the motion of a dipole in a uniformly rotating magnetic field $[S(t) = e^{i\omega t}]$. The model is exactly solvable and the physical observables are periodic functions of time. Autler and Townes[7] have considered the influence of a single monochromatic field [$S(t) = \cos\omega t$, with no counterrotating component as in the Rabi case]. In this model, the Floquet theorem applies because of the periodicity of the Hamiltonian and the physical observables can be shown to be quasiperiodic, although their precise calculation necessitates more or less lengthy numerical computations. However, at low intensities a perturbation approach allows us to reach analytical results.

The question of a bichromatic, or quasiperiodic, external perturbation has also received some attention. Indeed, Ho and Chu[8] have introduced what they called the many-mode Floquet theory to treat this problem. This leads to consideration of infinite-dimensional matrices with constant coefficients and therefore the truncation problem may be nontrivial. In any case, these studies have not contemplated the possibility of chaotic manifestations in this model. Chaos has been detected[9] in the Tavis-Cummings model[10] which consists of a set of two-level atoms interacting with a single-mode classical electromagnetic field. Below, we consider the effect of an external forcing on a single quantum system with two levels, and show that it may exhibit chaotic behavior. While the chaos in the Tavis-Cummings model has a lot in common with the classical Hamiltonian chaos with a few degrees of freedom, we do not expect such a similarity to show up in our problem. In the Tavis-Cummings model, by analogy with classical Hamiltonian chaos, one has coexistence of quasiperiodic behavior (= trajectories on Kolmogorov-Arnol'd-Moser tori) and chaotic behavior for the same values of the control parameters, depending on initial conditions. On the other hand, the chaos that we find seems to be independent of initial conditions for a given set of parameters.

As a first step, we investigate the time dependence of the observable quantities

$$A = |\psi_2|^2 - |\psi_1|^2,$$
$$B = i(\psi_2\psi_1^* - \psi_1\psi_2^*),$$
$$C = \psi_2\psi_1^* + \psi_1\psi_2^*.$$

The time evolution of A, B, C is governed by Bloch-type equations of motion:

$$\dot{A} = 2S(t)B,$$
$$\dot{B} = -2S(t)A - (\omega_1 - \omega_2)C,$$
$$\dot{C} = (\omega_1 - \omega_2)B.$$

The main difference from true Bloch equations is that the field $S(t)$ is externally imposed. We have computed numerically the time dependence of A, B, and C for two different cases: (i) $S(t) = g\cos\omega t$, i.e., a periodic perturbation (monochromatic light), and (ii) $S(t) = g\cos\omega t\cos\omega' t$, i.e., a quasiperiodic perturbation (bichromatic light). Our choice for the frequencies ω, ω' was $\omega = \frac{17711}{28657}$ and $\omega' = \frac{4637}{13313}$, both denominators being prime (thus making ω and ω' irrational for practical computations). As the perturbation in case (i) is periodic, Floquet theorem applies and a quasiperiodic behavior is expected for A, B, and C. This is in fact what can be observed in Fig. 1(a) for A (and similar results hold true for B and C). The dynamics is quite regular and the overall picture is compatible with quasiperiodicity. Case (ii), on the other hand, leads to a regular behavior for neither A, B nor C. In fact, Fig. 2(a) suggests that the dynamics in this case may be chaotic.

In order to render the characterization of the regularity of A more quantitative, we have computed the Fourier spectrum of discretized time series $A(t_i)$ in cases (i) and (ii). Figures 1(b) and 2(b) show the

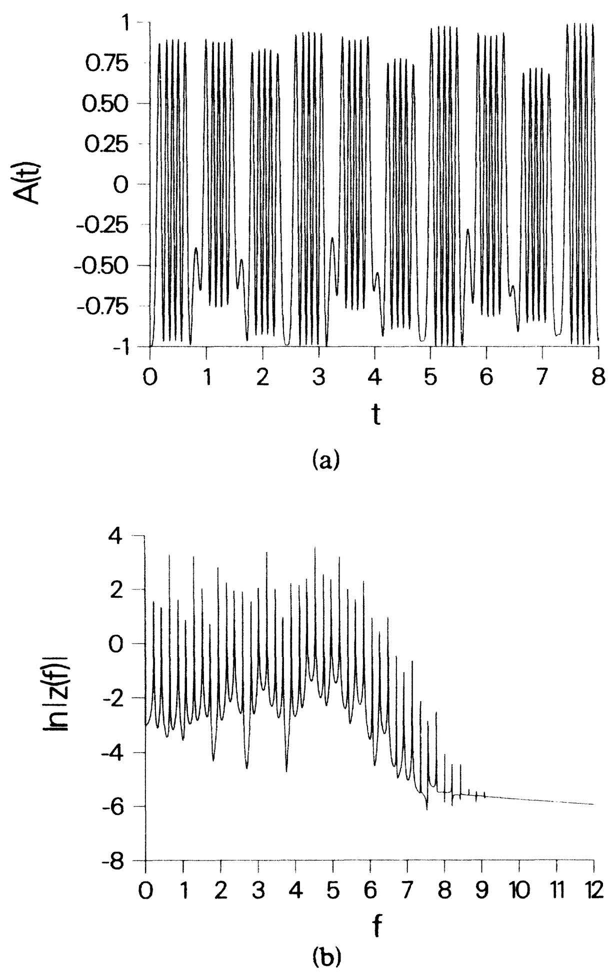

FIG. 1. (a) Time dependence of the observable A (difference in level population) as a function of time, in units of $2\pi/(\omega_1 - \omega_2)$ (with $\omega_1 - \omega_2 = 1$ here) for the monochromatic case ($g = 5$). (b) Power spectrum of the above in units of $(\omega_1 - \omega_2)/N$ with $N = 2^{15}$.

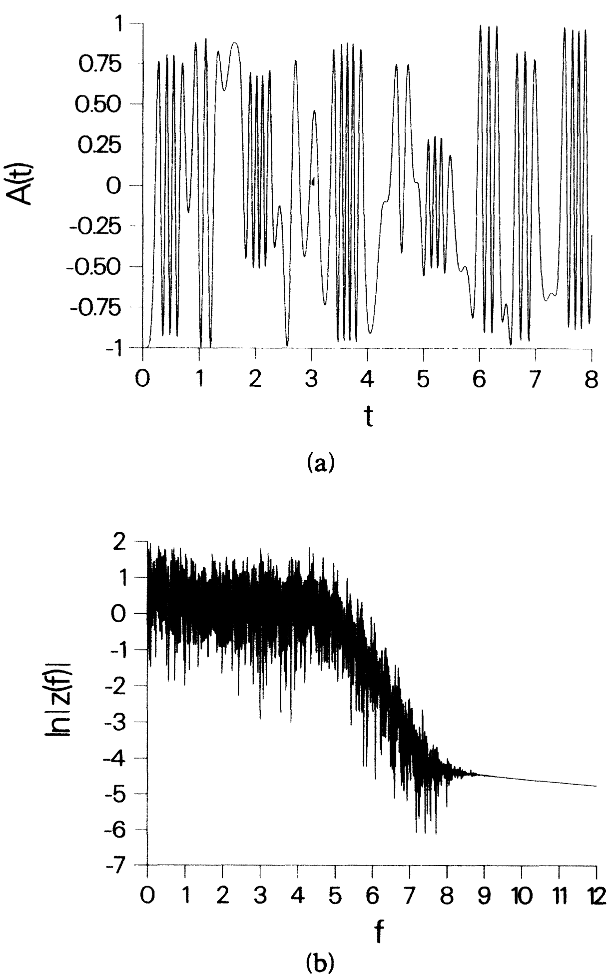

FIG. 2. Same as in Fig. 1 for the bichromatic case.

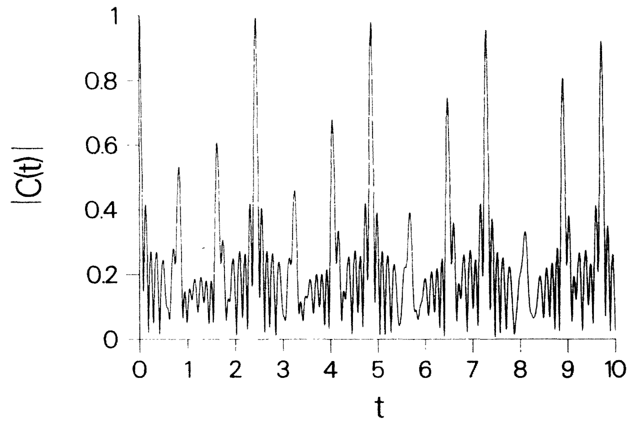

FIG. 3. Modulus of the autocorrelation function for the monochromatic case ($g = 5$).

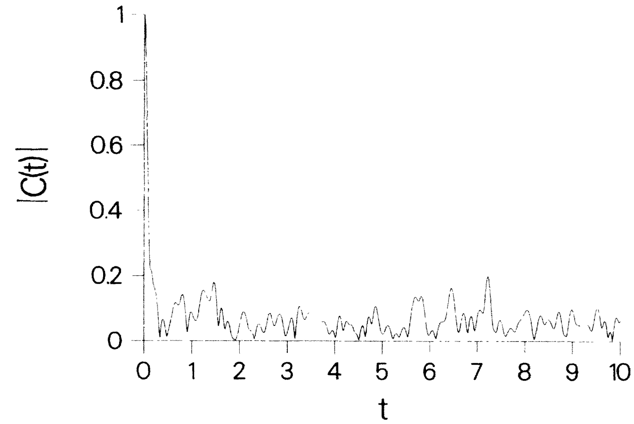

FIG. 4. Same as in Fig. 3 for the bichromatic case.

power spectra [logarithms of the moduli of the Fourier coefficients $Z(f)$, obtained through a fast Fourier-transform routine, using $N = 2^{15}$ points of the time series $A(t_i)$]. The difference between the two spectra is striking. Figure 1(b) shows a collection of well-defined peaks and the regularity in their distribution suggests the existence of just a few underlying fundamental frequencies: a typical spectrum for quasiperiodic behavior. Figure 2(b) shows a profusion of spectral lines of comparable amplitudes, which is considered as a signature of chaos. In fact, this spectrum tends to be continuous, once smoothed out over neighboring frequencies.

Having established the existence of a chaotic behavior in observable time-dependent quantities, we proceed to study another physical quantity which has proven to be of great help in the study of quantum chaos, namely the time-dependent autocorrelation function (ACF) for the wave function itself[11]:

$$C(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \psi_i^*(t) \psi_i(t+\tau) dt.$$

A rapidly decaying autocorrelation is a signature of chaos,[12] while a quasiperiodic solution of the system will reflect itself in a similarly quasiperiodic autocorrelation. In Figs. 3 and 4, we present our results for the autocorrelation functions (in fact the plots are in terms of their moduli) as functions of time for cases (i) (Fig. 3) monochromatic, and (ii) (Fig. 4) bichromatic perturbation. As expected, case (i) leads to a nondecaying ACF, a feature independent of the value of the coupling constant g. In the case of a bichromatic perturbation, a fast decrease of the ACF is obtained. The decay of the ACF depends on g. This can be made clearer by considering the ratio of the highest maximum of the ACF that we have obtained, besides the one at $\tau = 0$, to the value of the ACF at $\tau = 0$ (which can be arbitrarily normalized to 1). In Fig. 5, we plot this ratio R vs g. This ratio decreases fairly rapidly as g increases. Beyond $g \simeq 4$ the decrease proceeds, but at a slower rate. This reflects (loosely speaking) the fact that chaos gets more pronounced as g increases and becomes well established beyond $g \simeq 4$. Thus this second indicator of chaos points towards the same conclusion as the first: The two-level system shows a chaotic behavior when illuminated by a bichromatic external source (while the use of monochromatic light results in the well-known quasiperiodic behavior of the Autler-Townes model).

Another well-known indication of chaos is the existence of Lyapunov number(s) representing the amplification of initial fluctuations. In the present case, it does not seem possible to have such positive Lyapunov numbers. Actually, the equations of motion are linear, and a positive Lyapunov number would imply the existence of solution growing exponentially in average, which is impossible because of the conserva-

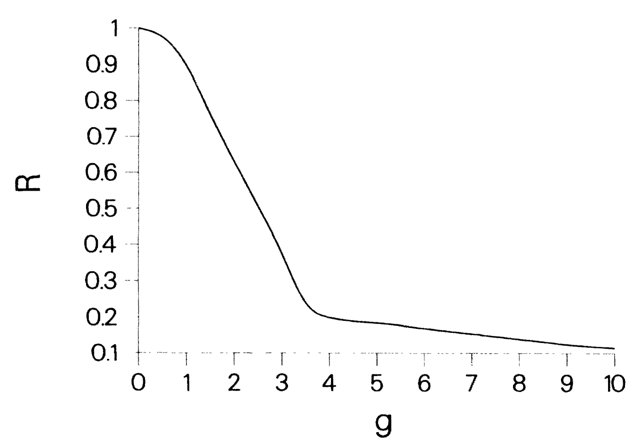

FIG. 5. Highest maximum (other than for $\tau = 0$) of the ACF for the bichromatic case as a function of the perturbation strength g.

tion of the norm.

As stated at the outset, experimental realizations of our system appear feasible. Thus NMR experiments on spin-$\frac{1}{2}$ systems with a two-frequency perpendicular magnetic field $H_\perp$ could provide some evidence on the existence of chaotic Rabi oscillations.[13] Similarly, a study of molecular or atomic processes under the influence of intense bichromatic radiation could show chaotic "Autler-Townes"-type oscillations, similar to those described in this paper. In this last case, indeed, one should include in the theoretical picture—to make it more realistic—the loss of coherence in oscillations due to spontaneous radiative decay. At this stage, several extensions of this work to multilevel systems appear worthwhile. One of the most promising is the study of a three-level system with Planck frequencies ω_1, ω_2, and ω_3, similar to the one of Ho and Chu.[14] In this case, there are two resonant external excitations at frequencies $\omega_2-\omega_1$ and $\omega_3-\omega_1$ and the quasiperiodicity results, in some sense, from the two different frequencies of Rabi oscillations. The possibility for chaotic behavior in this system is currently under investigation and will be the subject of a future publication.

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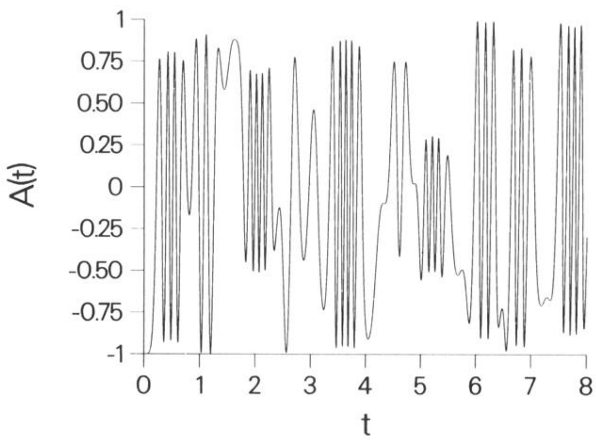
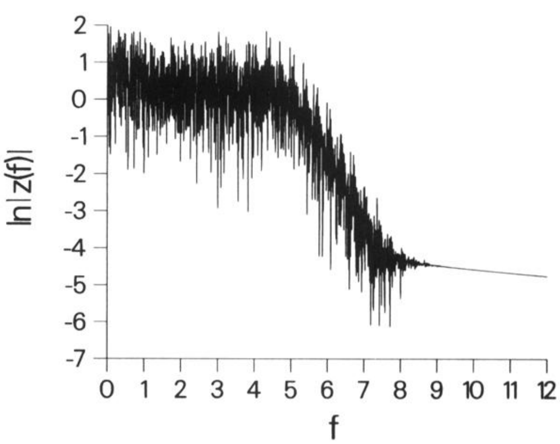


(a)



(b)

FIG. 2. Same as in Fig. 1 for the bichromatic case.