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Fundamental work cost of quantum processes

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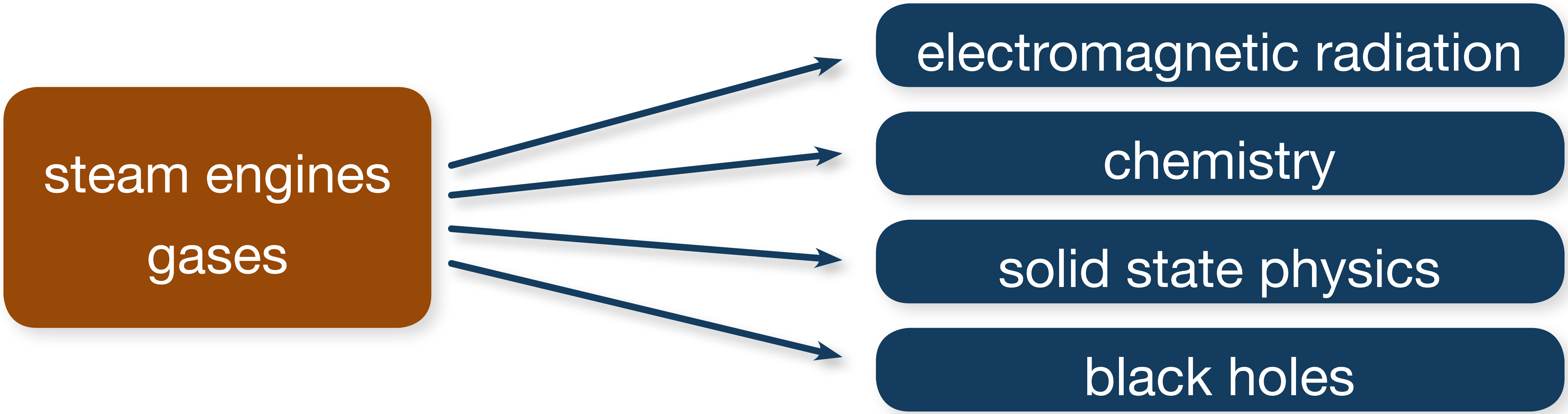
QIP 2018, Delft, January 2018


steam engines
gases
electromagnetic radiation
chemistry
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Maxwell's demon

anomalous heat flows

Jennings & Rudolph, PRE, 2010

side information

del Rio *et al.*, Nature, 2011

?

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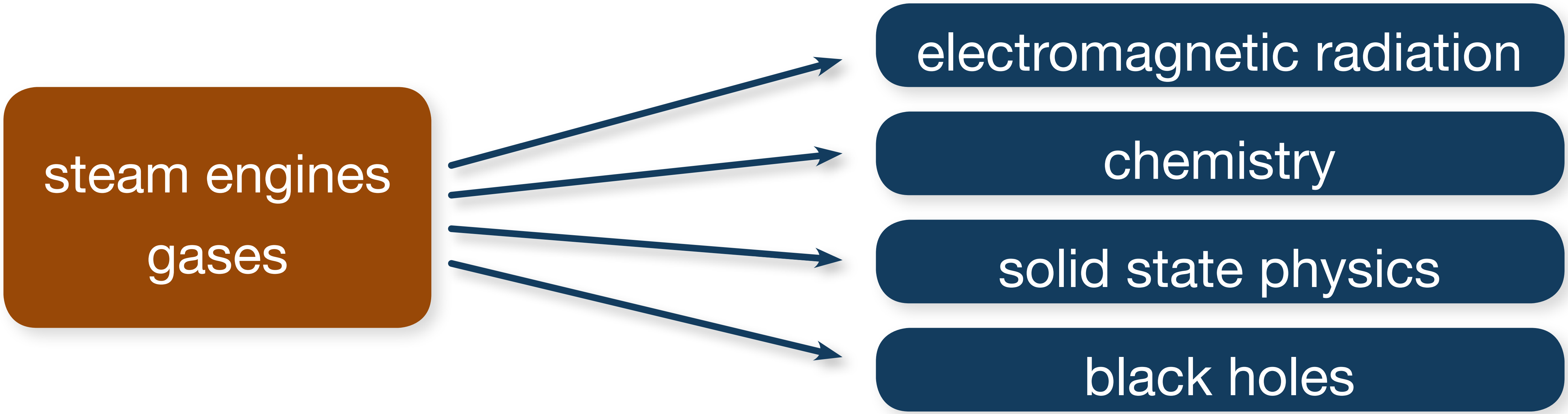
✓ actually OK, just need to be careful



Maxwell's demon

anomalous heat flows

Jennings & Rudolph, PRE, 2010

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actually OK, just need to be careful

- **What is the most general formulation of thermodynamics?**

steam engines
gases

- electromagnetic radiation
- chemistry
- solid state physics
- black holes

Maxwell's demon

anomalous heat flows

actually OK, just
need to be careful

Jennings & Rudolph

side inform

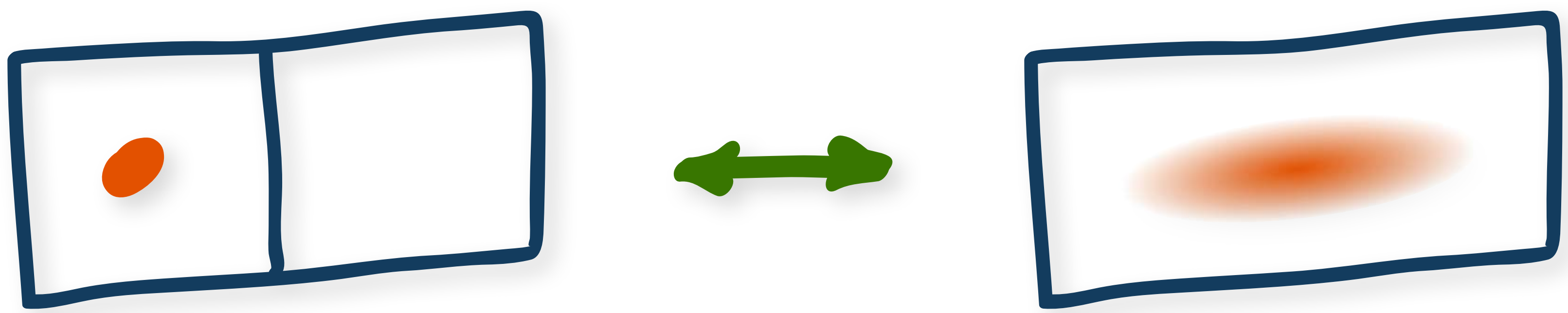
Idea: role of information

f

del Rio *et al.*,

Information and Thermodynamics

1 bit of information can be traded for $kT\ln 2$ work



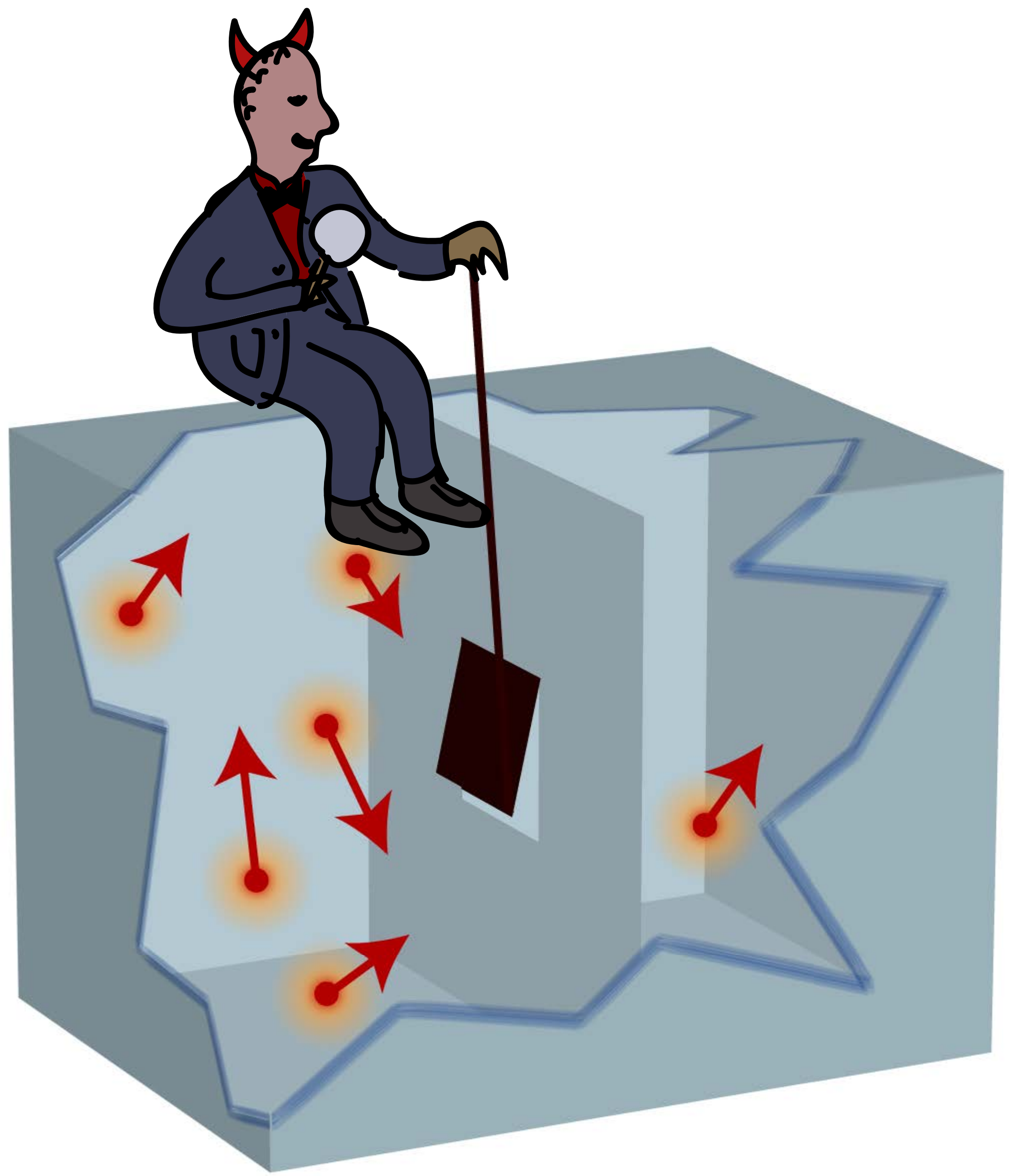
Szilárd engine

Szilárd, 1929

Landauer: Irreversible information processing incurs thermodynamic cost

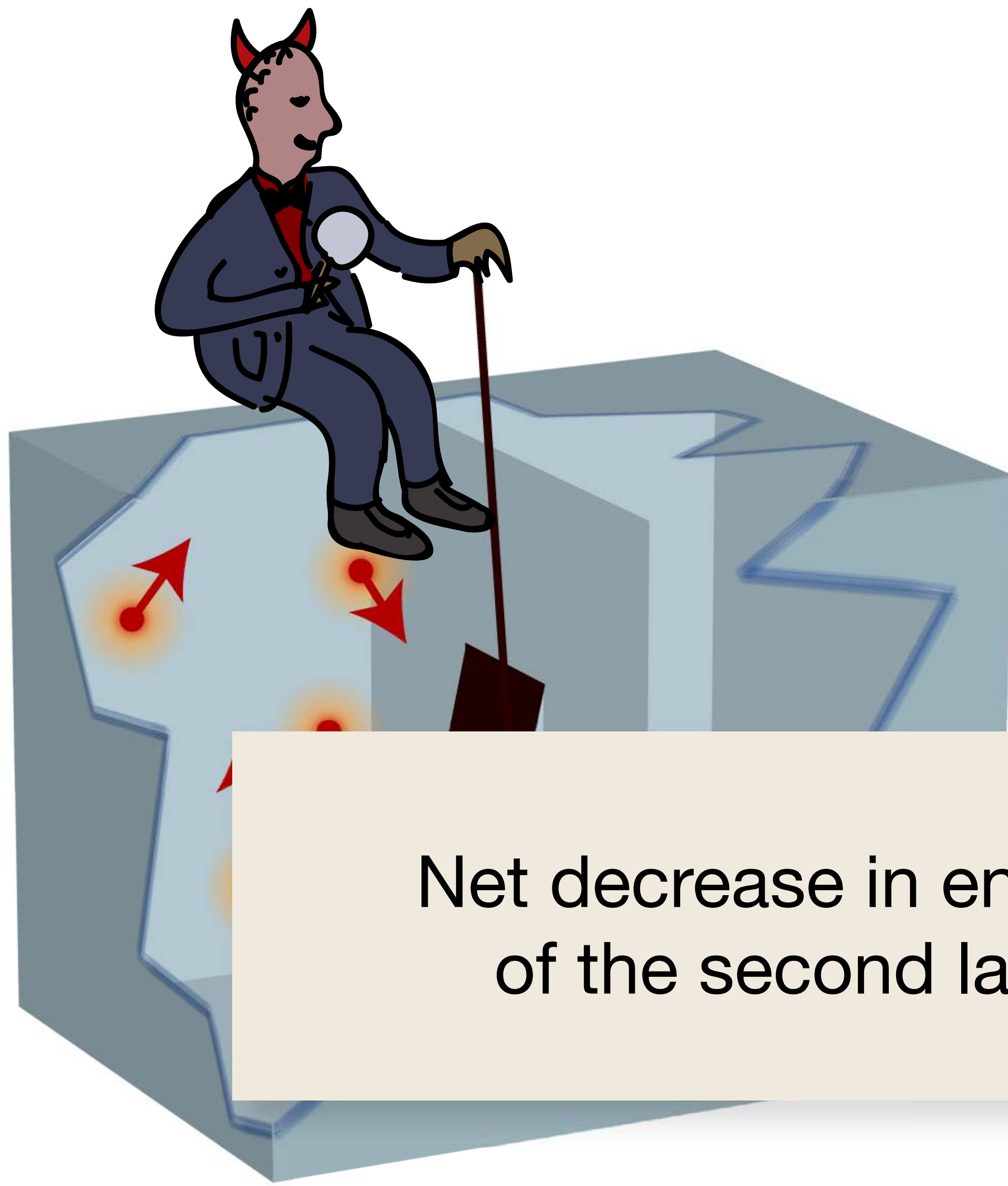
Landauer, 1961
Bennett, 1982, 2003

Maxwell’s Demon



Demon lets particles go
from right to left only

Maxwell's Demon



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Net decrease in entropy of gas $\longrightarrow$ violation
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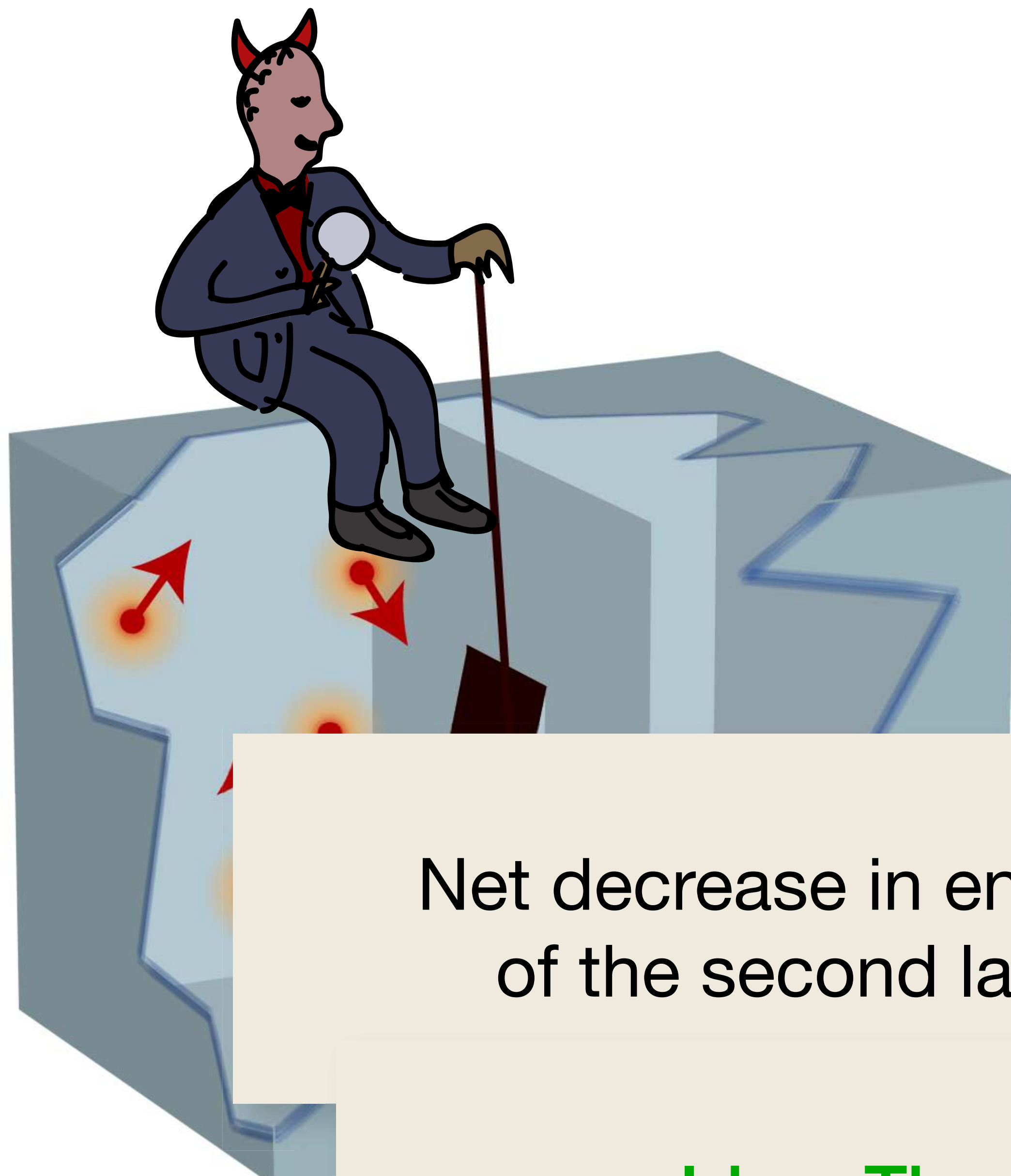
Maxwell's Demon

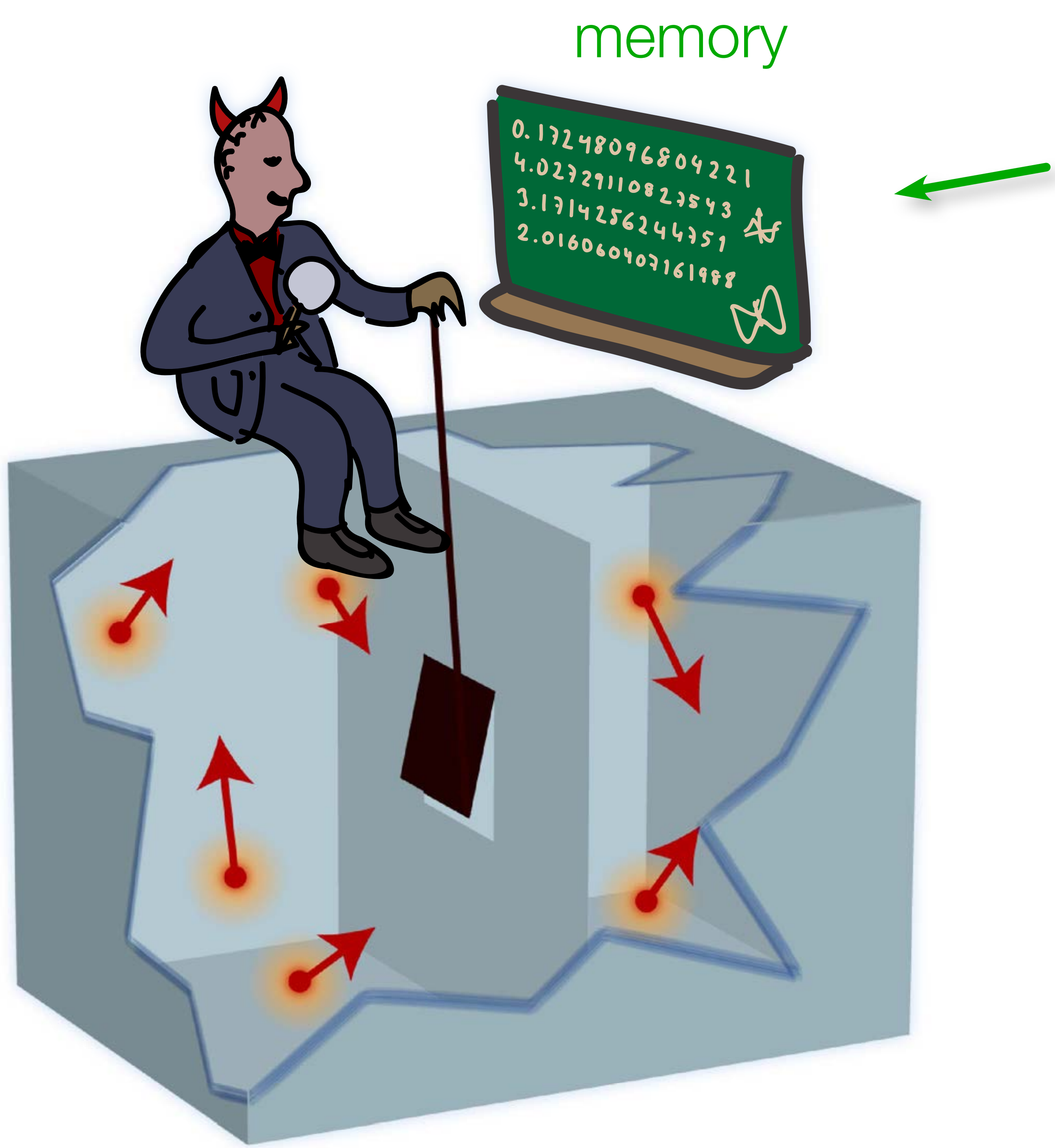


Demon lets particles go from right to left only

Net decrease in entropy of gas $\longrightarrow$ violation of the second law of thermodynamics ?

Idea: The demon has access to microscopic information

Maxwell's Demon

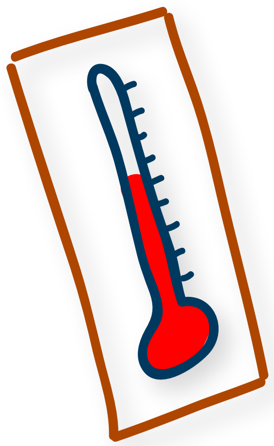


The demon stores the measurement results

Resetting this memory costs work!

Landauer, 1961
Bennett, 1982, 2003

Resource Theory of thermal operations



- Allowed any ancilla in a Gibbs state

$$\gamma_B = e^{-\beta H_B} / \operatorname{tr}(e^{-\beta H_B})$$

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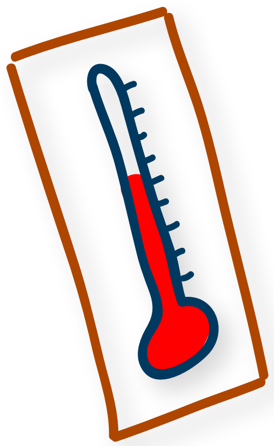

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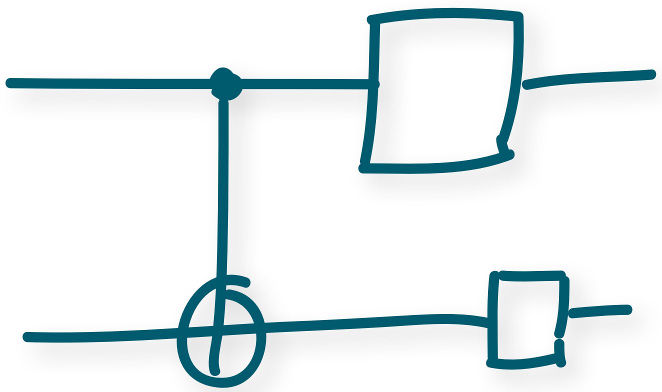
$$[U, H_{\text{total}}] = 0$$



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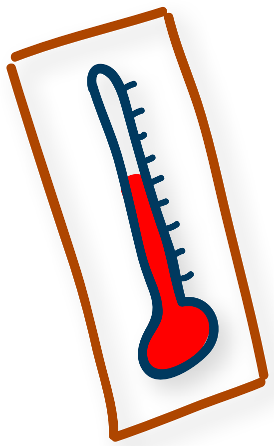

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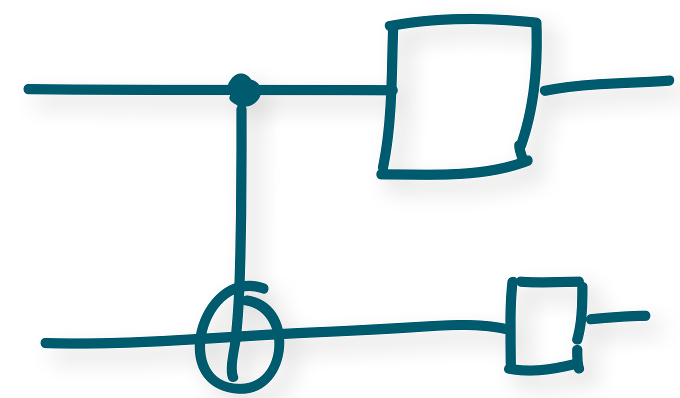


- Allowed to discard any system

Known Results

- Necessary and sufficient conditions for $\rho \to \sigma$ (thermo-majorization, block-diagonal states)

  Horodecki & Oppenheim, Nat. Comm. 2013

- Conversion rates $\rho^{\otimes m} \leftrightarrow \sigma^{\otimes n}$

  Brandão *et al.*, PRL, 2013

- Rényi-α entropies monotones: “second laws”

  Brandão *et al.*, PNAS, 2015

- Generalized thermodynamic baths

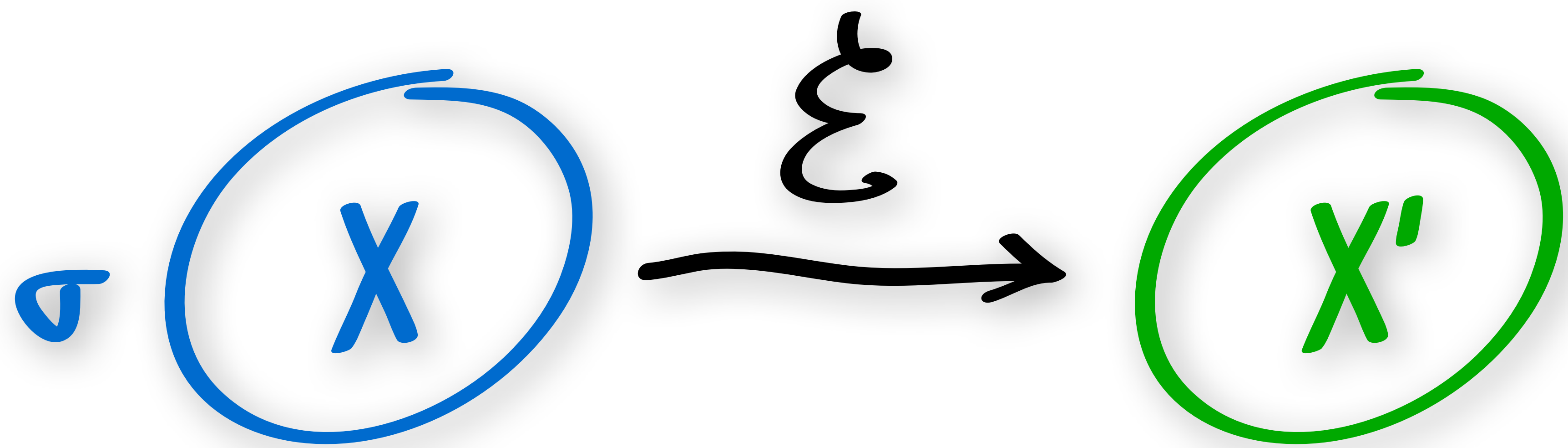
  Yunger Halpern & Renes, PRE, 2016 , ...

- Catalytical transformations, correlations ...

  Ng *et al.*, NJP, 2015; Lostaglio *et al.*, PRL, 2015 ...

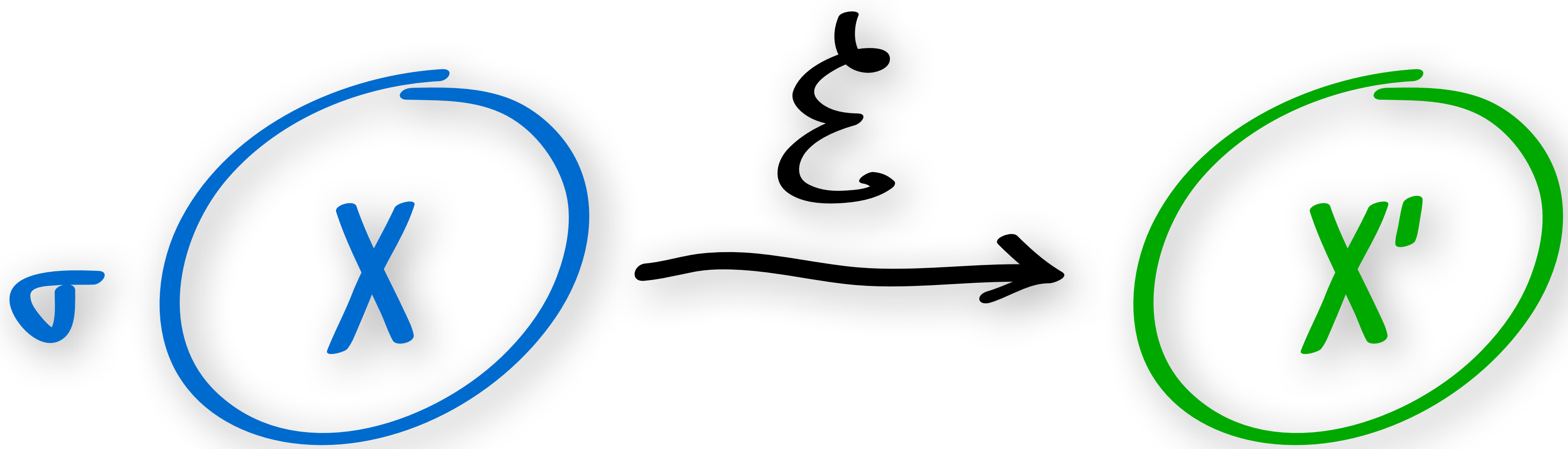
...

Thermodynamic cost of any process?



- mapping of input states to output states
  - → AND, XOR, … gate
  - → any classical or quantum computation
  - → any physical process (completely positive, trace-preserving map)

Thermodynamic cost of any process?



- mapping of input states to output states

→ AND, XOR gate

→ any c

→ any p
prese

Fundamental thermodynamic limit
to the cost of implementing $\mathcal{E}$?

Our approach

A restriction on what we can do

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- **Free operations must preserve the thermal state**

(most generous set of maps)

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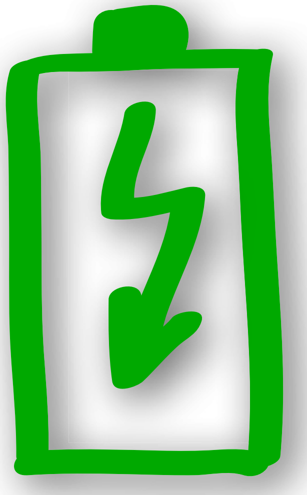
A resource which we can use to overcome the restriction



Our approach

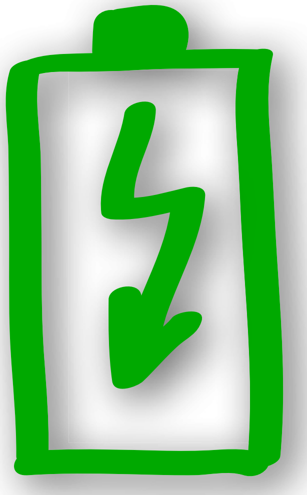
A restriction on what we can do

- **Free operations must preserve the thermal state**

(most generous set of maps)

A resource which we can use to overcome the restriction

- **Battery system**



Framework (2)

Γ

To each system of interest S is associated an operator $\Gamma_S \geqslant 0$

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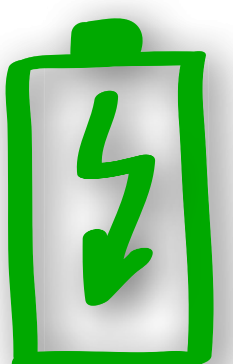

Information battery

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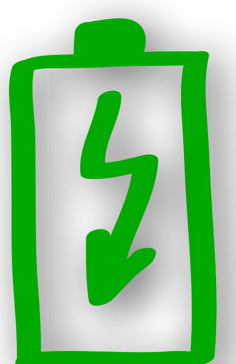


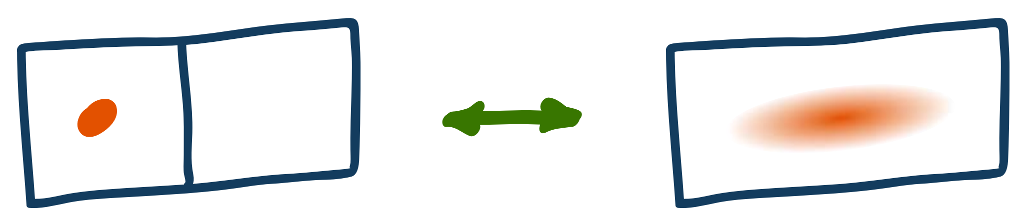
Information battery

Large family of battery models are equivalent

Information Battery

$\underbrace{|0\rangle\ |0\rangle\ |0\rangle \ldots |0\rangle}_{n-\lambda \text{ pure qubits}}\ \underbrace{\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2} \ldots \tfrac{1}{2}}_{\lambda \text{ maximally mixed qubits}}$

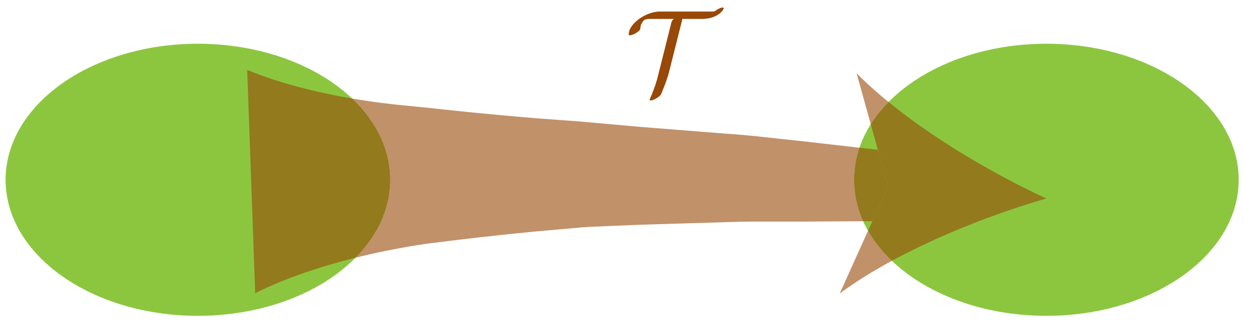
"stored work"



Landauer/Szilárd:
1 pure qubit = $kT \ln 2$ work

$H = 0 \longrightarrow \Gamma = 1$

Step #1: Limit for an exact process



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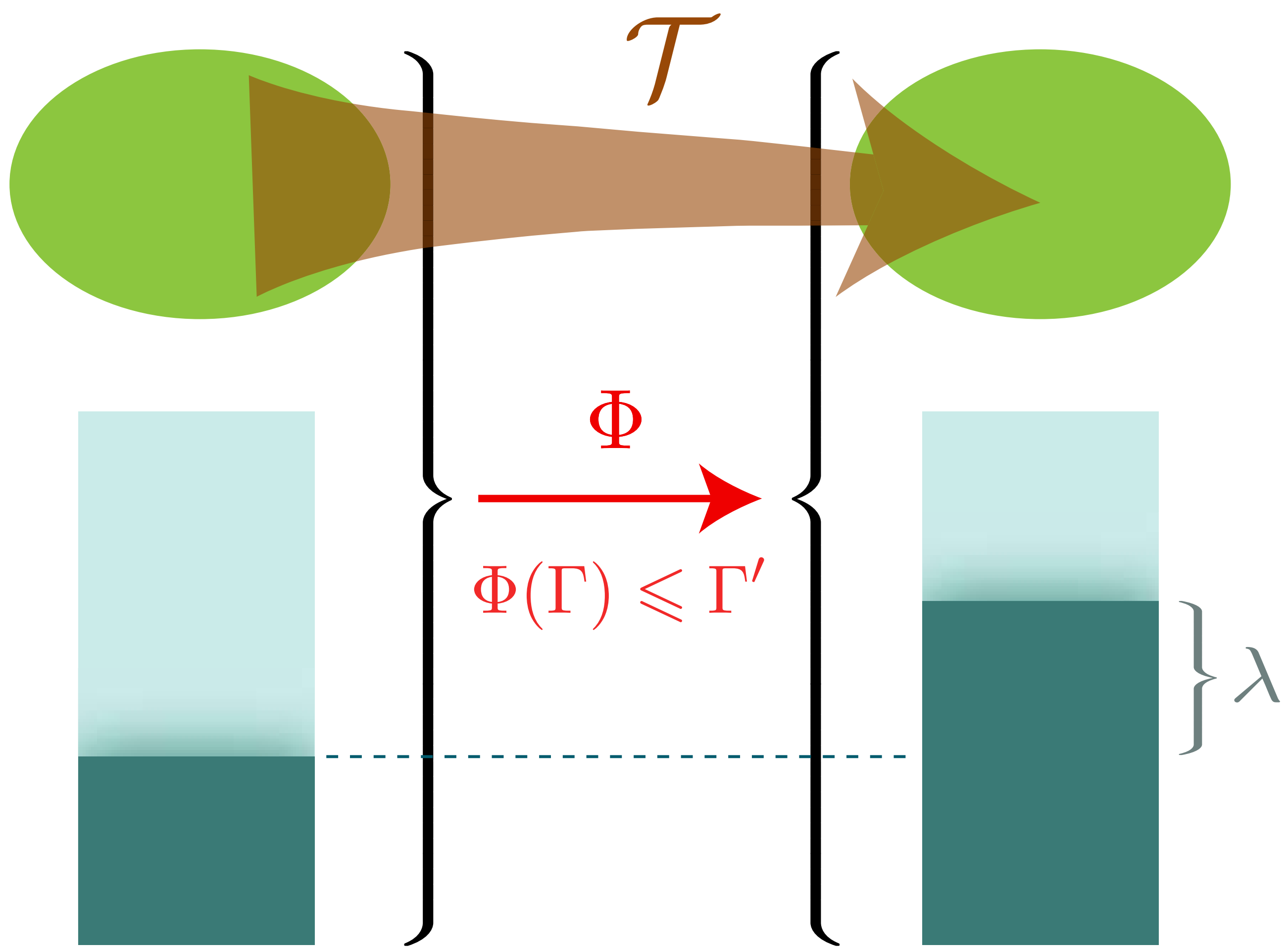

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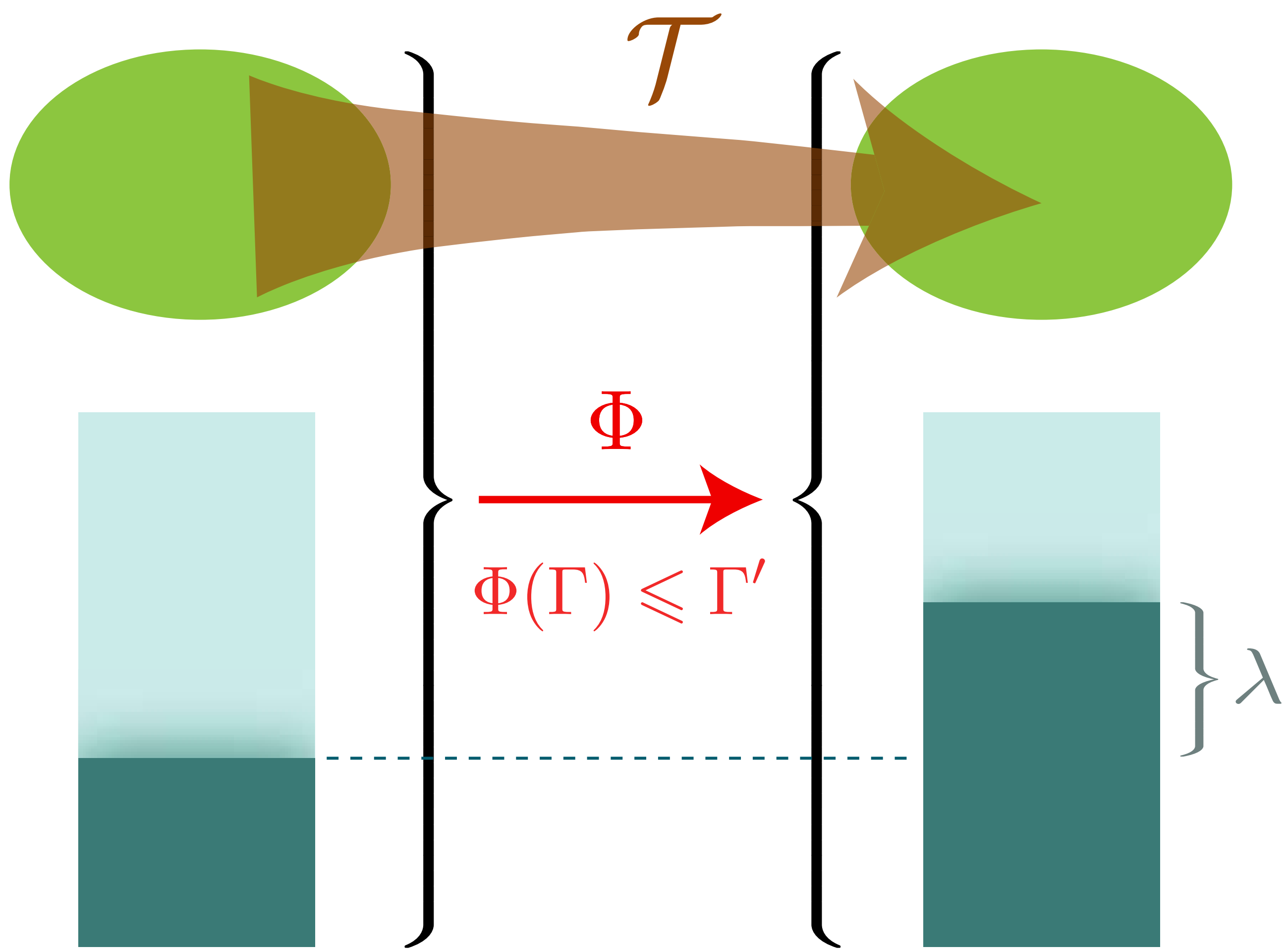

$$\exists\ \Phi :$$
$$\Phi(\Gamma_{\mathrm{tot}}) \leqslant \Gamma'_{\mathrm{tot}}$$
$$\Leftrightarrow$$
$$\mathcal{T}(\Gamma) \leqslant 2^{-\lambda}\,\Gamma'$$

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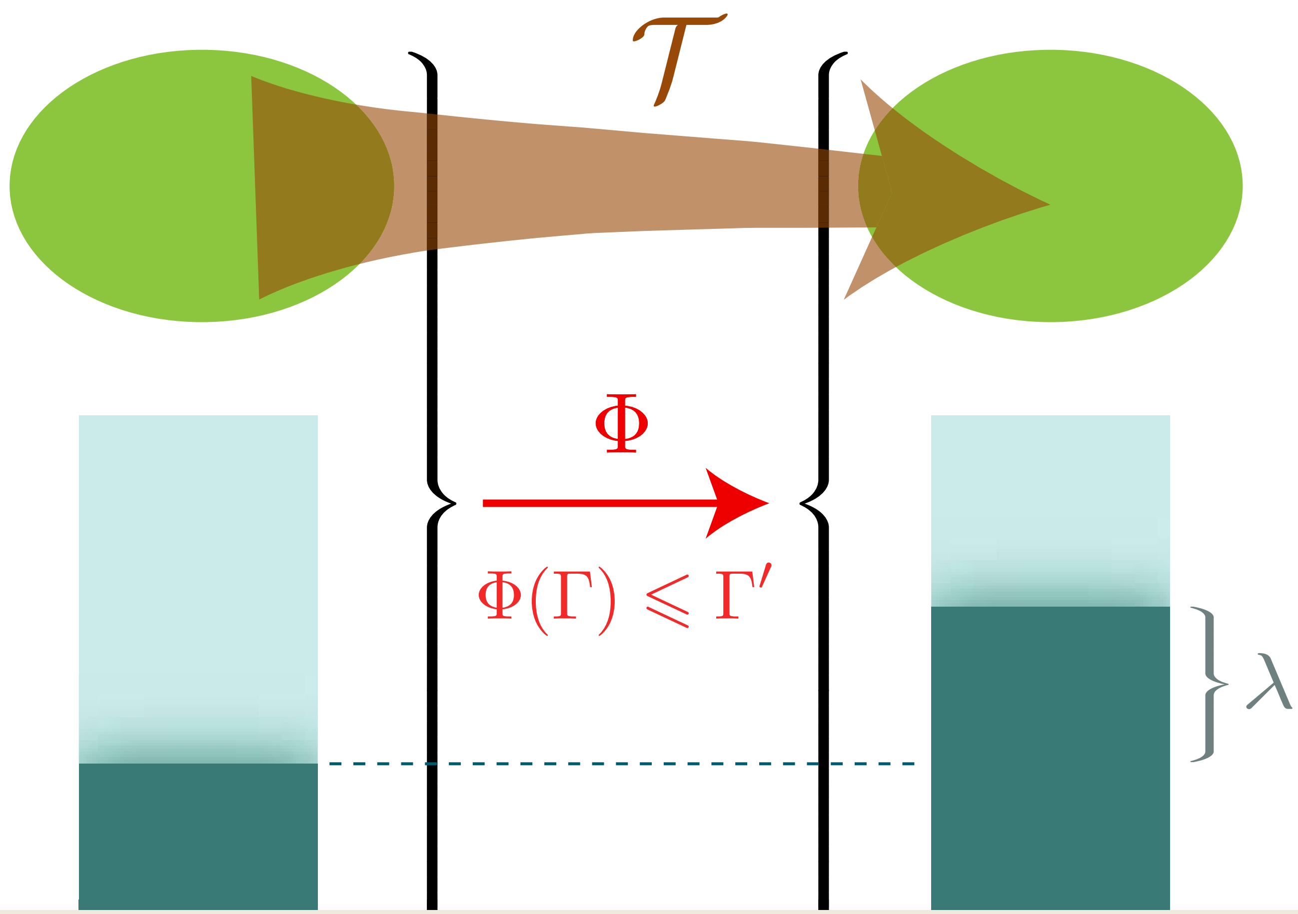

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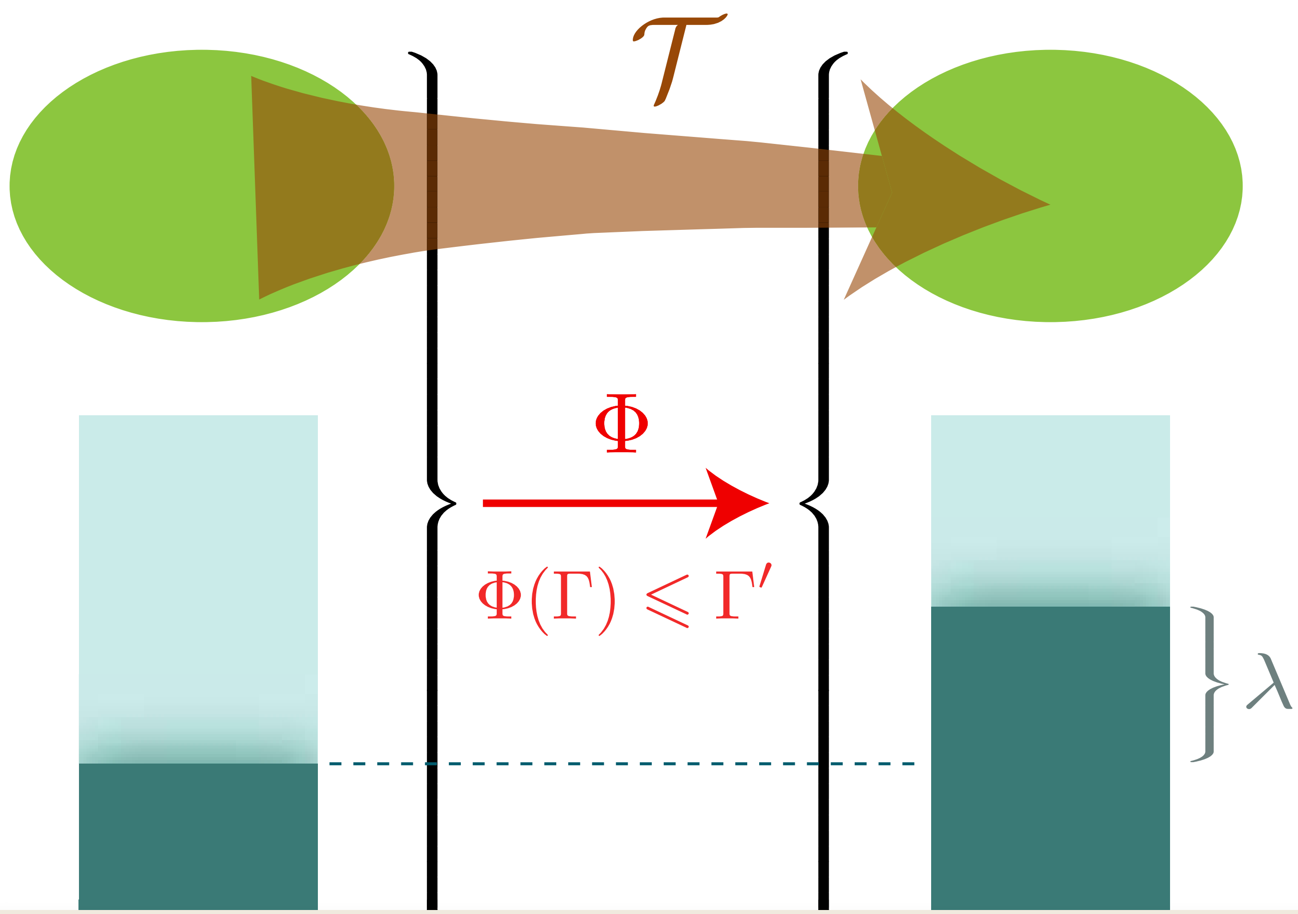

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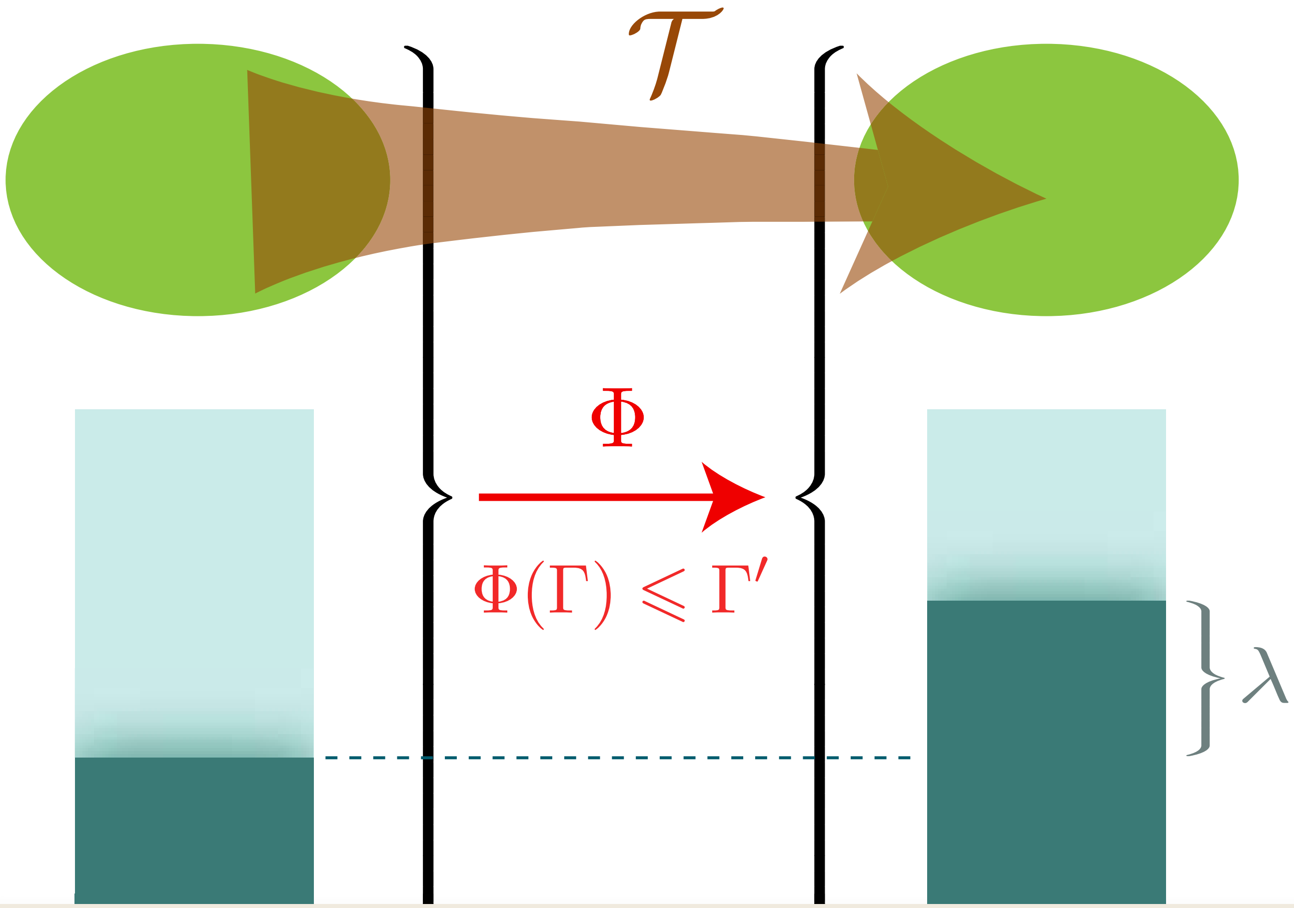

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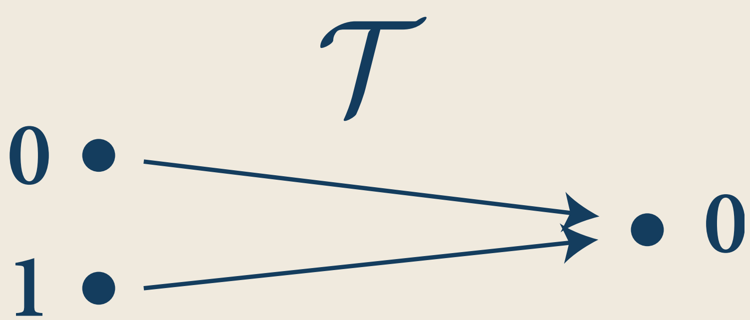
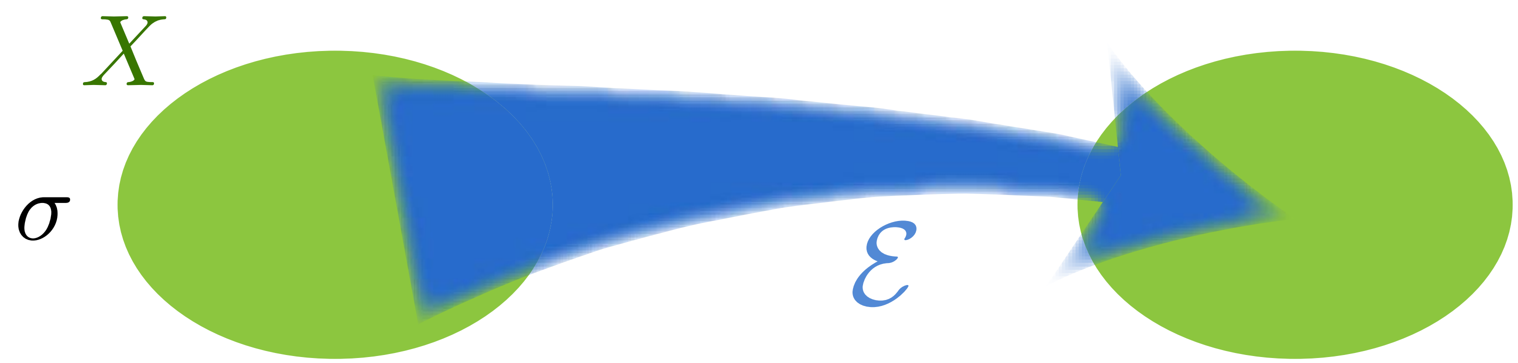


$\mathcal{T}(\mathbb{1}) = 2|0\rangle\langle 0| \leqslant 2\,\mathbb{1}$

$\Rightarrow \quad \lambda \leqslant -1$

Step #2: Optimize effective process



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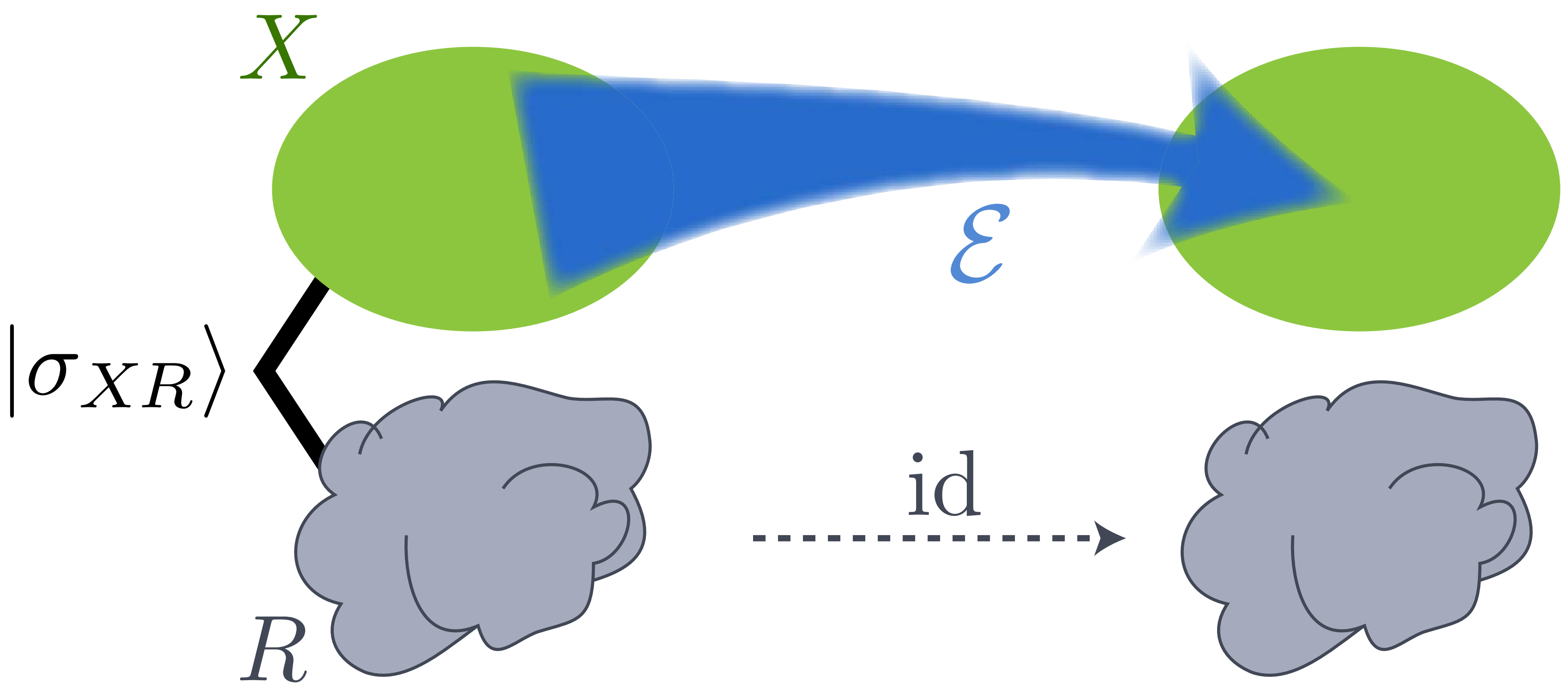

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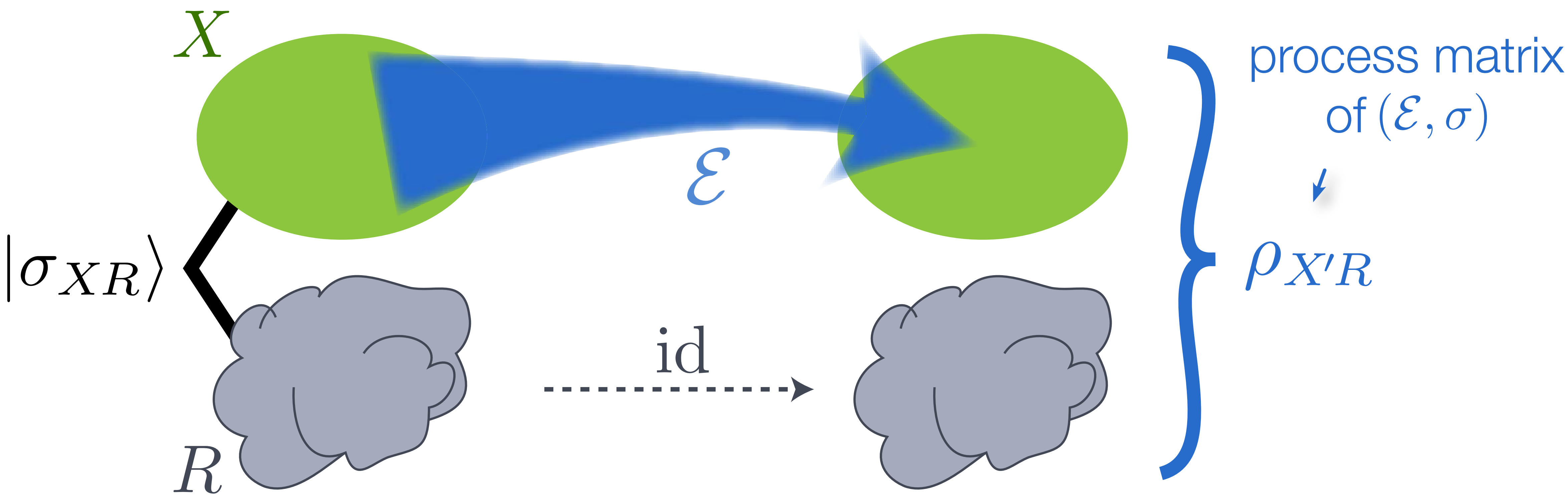

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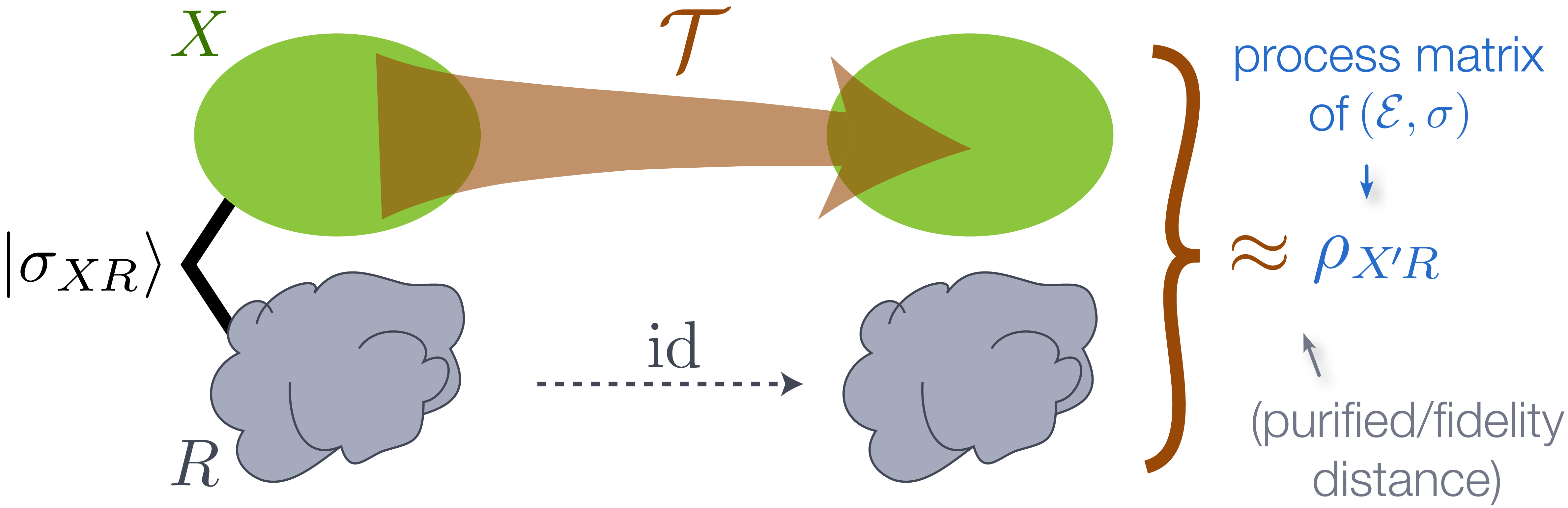

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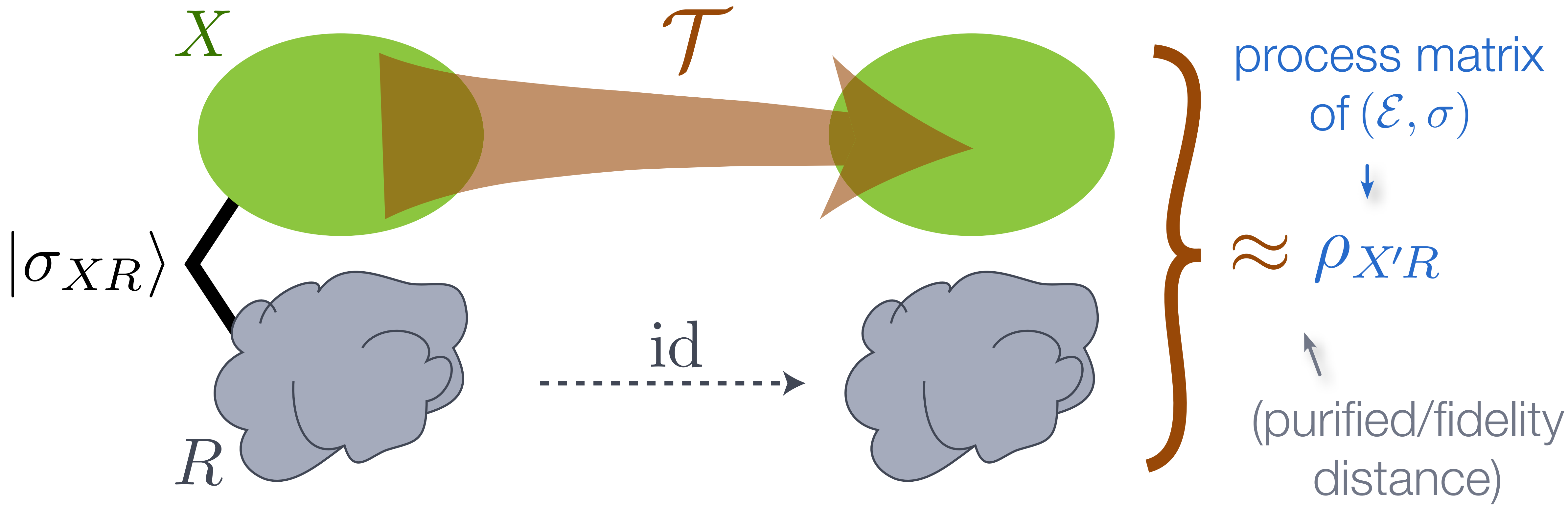
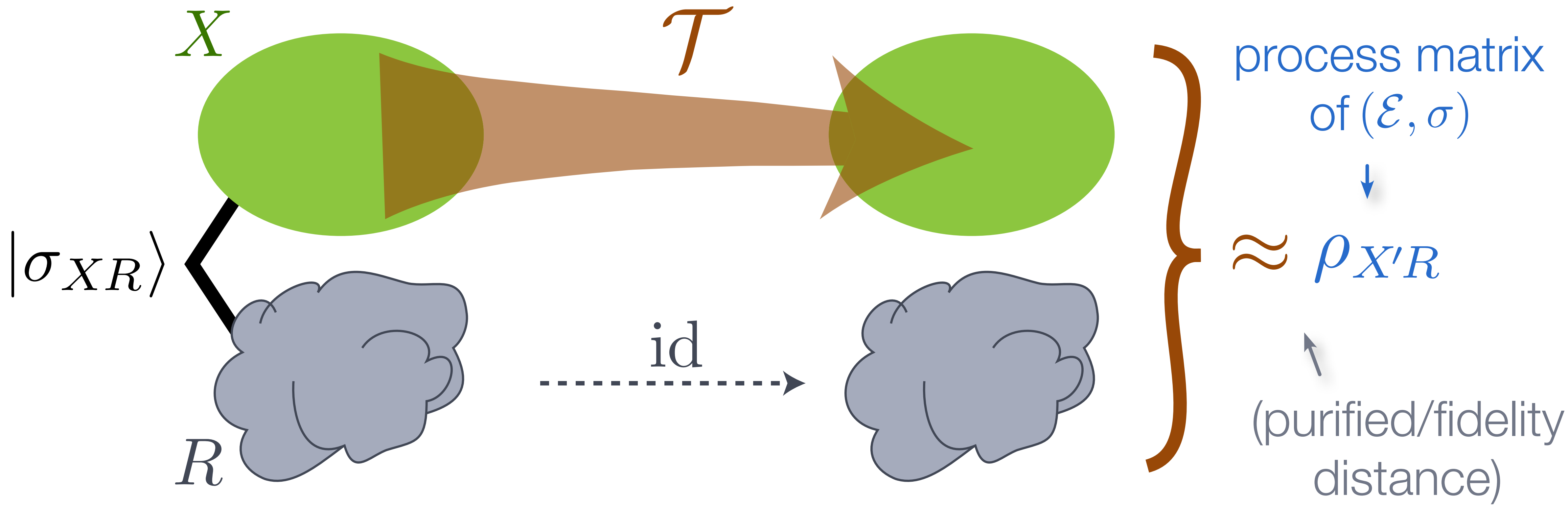


Step #1

limit for specific $\mathcal{T}$ = $\max_{\lambda:\ \mathcal{T}(\Gamma) \leqslant 2^{-\lambda}\,\Gamma} \lambda$

Step #2: Optimize effective process



Step #2 **Step #1**

fundamental limit = $\max\limits_{\mathcal{T}(\sigma_{XR}) \approx \rho_{X'R}} \quad \max\limits_{\lambda:\ \mathcal{T}(\Gamma) \leqslant 2^{-\lambda}\,\Gamma} \quad \lambda$

Results

Step #2 **Step #1**

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Results

Steps #1 & #2

fundamental limit = $$\max_{\substack{\mathcal{T}\ \text{c.p. tr.-noninc.},\ \lambda \\ \mathcal{T}(\Gamma)\leqslant 2^{-\lambda}\Gamma' \\ \mathcal{T}(\sigma_{XR})\approx\rho_{X'R}}} \lambda$$

Results

$$\hat{D}^{\epsilon}_{X\to X'}(\rho_{X'R}\,\|\,\Gamma_X,\Gamma'_{X'}) \;=\; \max_{\substack{\mathcal{T}\text{ c.p. tr.-noninc., }\lambda\\ \mathcal{T}(\Gamma)\leqslant 2^{-\lambda}\Gamma'\\ \mathcal{T}(\sigma_{XR})\approx\rho_{X'R}}} \lambda$$

coherent relative entropy

Ultimate maximum extractable work for implementing a map with process matrix close to $\rho_{X'R}$ =

$$kT\ln(2)\cdot\hat{D}^{\epsilon}_{X\to X'}(\rho_{X'R}\,\|\,\Gamma_X,\Gamma'_{X'})$$

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semidefinite program ✓

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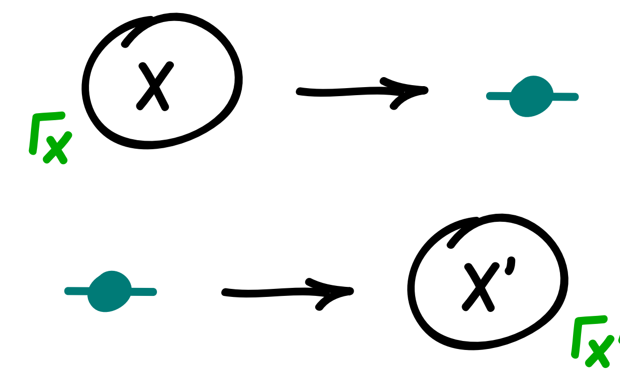
Coherent relatve entropy: Special cases

"relative"

$$\hat{D}_{X\to\varnothing}(\rho_X \,\|\, \Gamma_X, 1) = D_{\min}(\rho_X \,\|\, \Gamma_X)$$

$$\hat{D}_{\varnothing\to X'}(\rho_{X'} \,\|\, 1, \Gamma_{X'}) = -D_{\max}(\rho_{X'} \,\|\, \Gamma_{X'})$$



Datta, IEEE TIT (2009)
Åberg, Nat Comm (2013)
Horodecki & Oppenheim, Nat Comm (2013)

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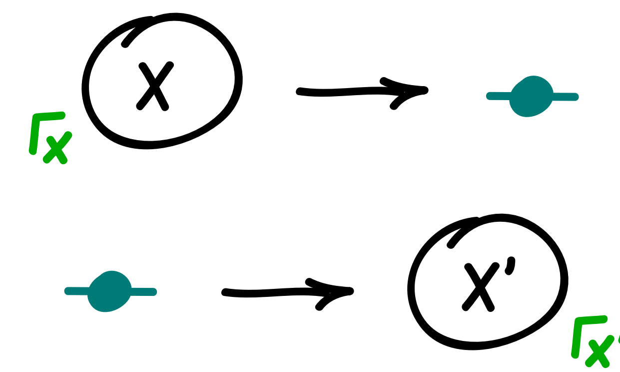

Datta, IEEE TIT (2009)
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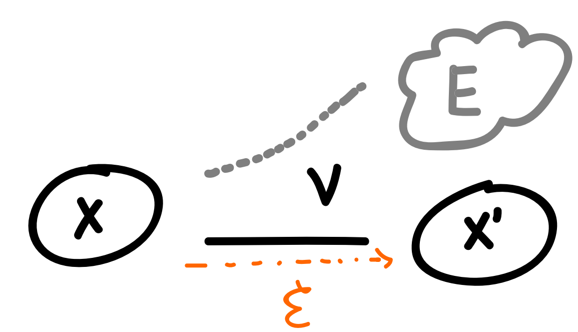
"coherent" ("conditional")

$$\hat{D}^{\epsilon}_{X\to X'}(\rho_{X'R} \,\|\, \mathbb{1}_X, \mathbb{1}_{X'}) = -\hat{H}^{\epsilon}_{\max}(E \,|\, X')_{\rho}$$



for pure $|\rho\rangle_{X'R_X E}$

Examples (with trivial Hamiltonian)

"Pure information processing" — no internal energy

$$\text{cost} = kT \ln 2 \cdot H_{\max}^{\epsilon}(E \mid X')_{\rho}$$

discarded information output

PhF, Dupuis, Oppenheim, Renner, Nat Comm, 2015

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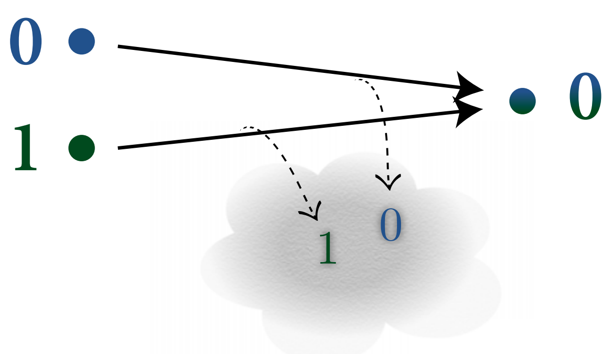
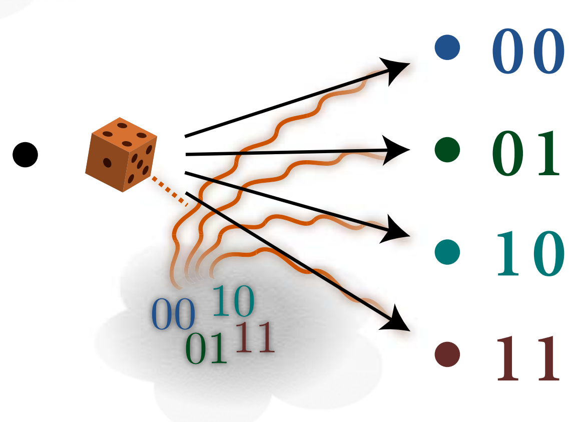
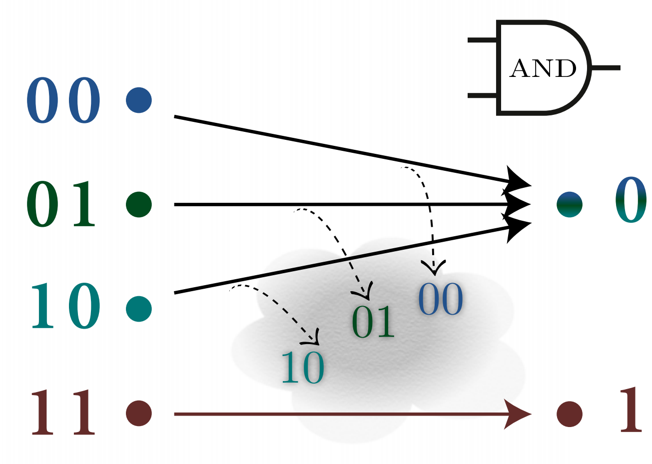


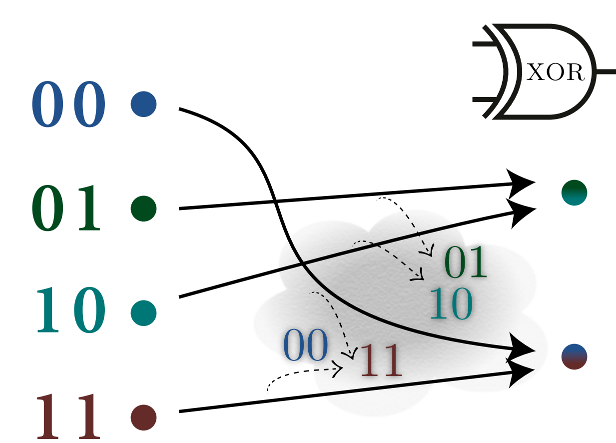
cost: 1 bit



extract: 2 bits



cost:
$\log(3) \approx$ 1.6 bits



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PhF, Dupuis, Oppenheim, Renner, Nat Comm, 2015

The Coherent Relative Entropy

$$\hat{D}^{\epsilon}_{X\to X'}(\rho_{X'R} \,\|\, \Gamma_X, \Gamma_{X'})$$

→ Data processing inequality

→ Chain rule

→ Asymptotic equipartition property: For many independent copies $\to D(\rho_X \,\|\, \Gamma_X) - D(\rho_{X'} \,\|\, \Gamma_{X'})$

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The coherent relative entropy:

- **measure of information**
- **has desirable properties**
- **reduces to known special cases**

Battery models

Define $\tau(P) = \dfrac{P\Gamma P}{\mathrm{tr}(P\Gamma)}$ for $[P, \Gamma] = 0$

For $\tau(P) \to \tau(P')$:

$$\hat{D}_{X\to X'}(\rho_{X'R} \,\|\, \Gamma, \Gamma') = \log \mathrm{tr}(P'\Gamma) - \log \mathrm{tr}(P\Gamma)$$

- **Reversibly interconvertable**

- **Some common battery models equivalent** (information battery, wit, weight)
- **Battery states are robust to smoothing** (no need to smooth battery states)

Brandão et al., PNAS (2015)

Emergence of Macro Thermodynamics

For a certain class of states (e.g. microcanonical):

$$\bar{D}^{\epsilon}_{X \to X'}(\rho_{X'R_X} \,\|\, \Gamma_X, \Gamma_{X'}) = \Lambda(\rho_R) - \Lambda(\rho_{X'})$$

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derives from a potential!

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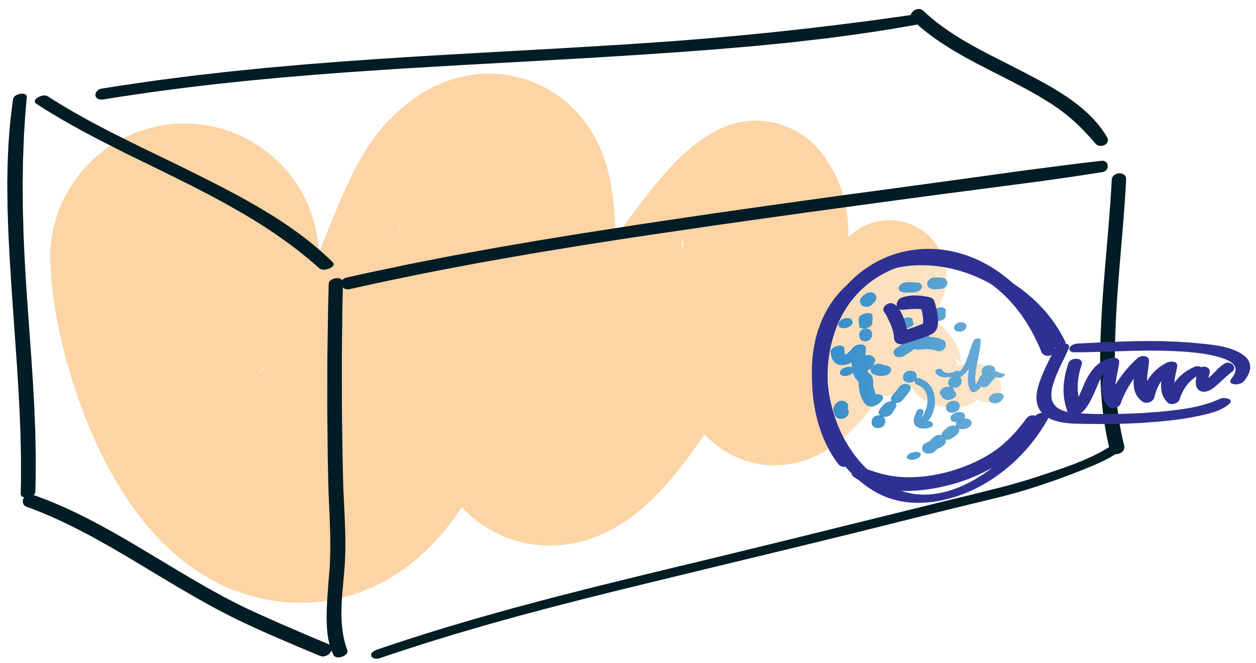
derives from a potential!

free $\rho \to \rho + d\rho$ within class $\to d\Lambda \leqslant 0$

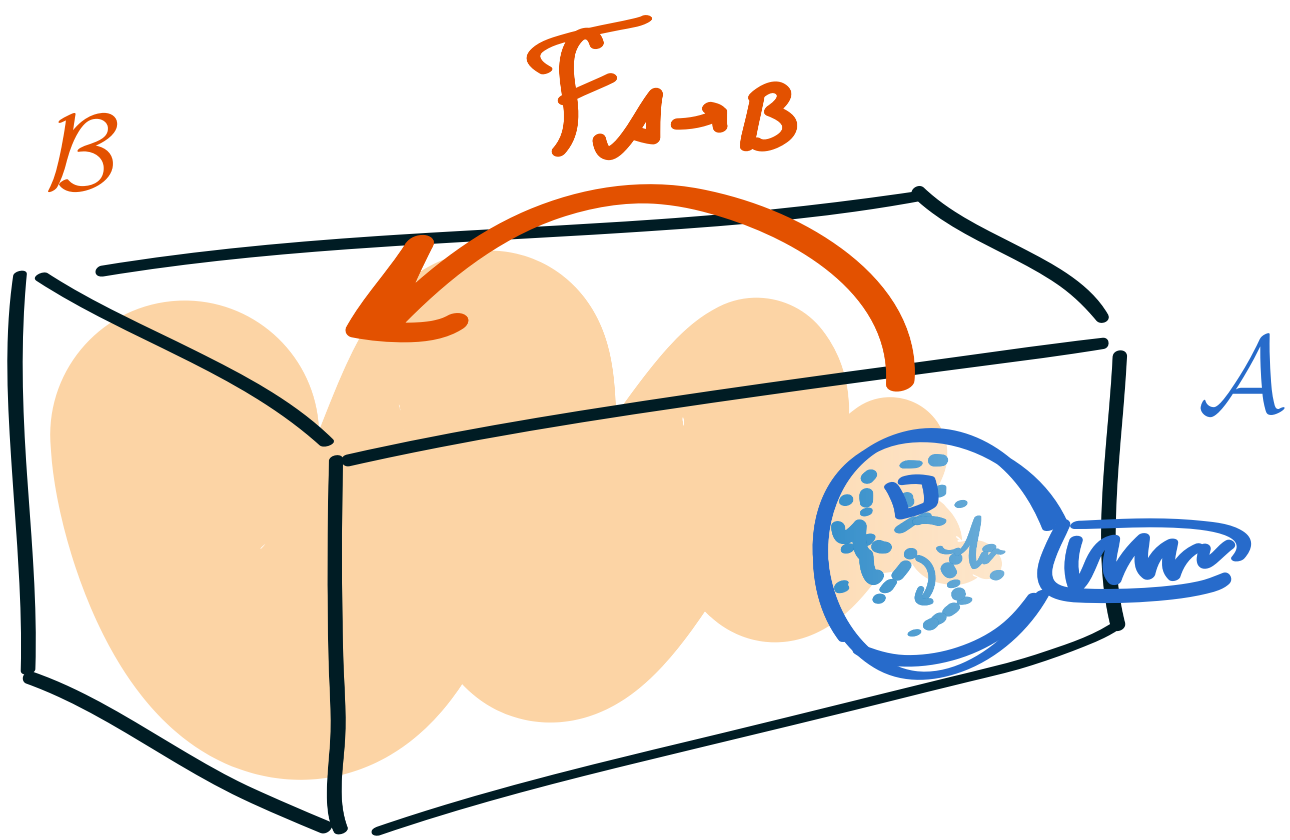
- isolated system: $\Lambda = -S$
- contact with heat bath: $\Lambda = \beta F$

- no i.i.d. assumption

Observers in Thermodynamics

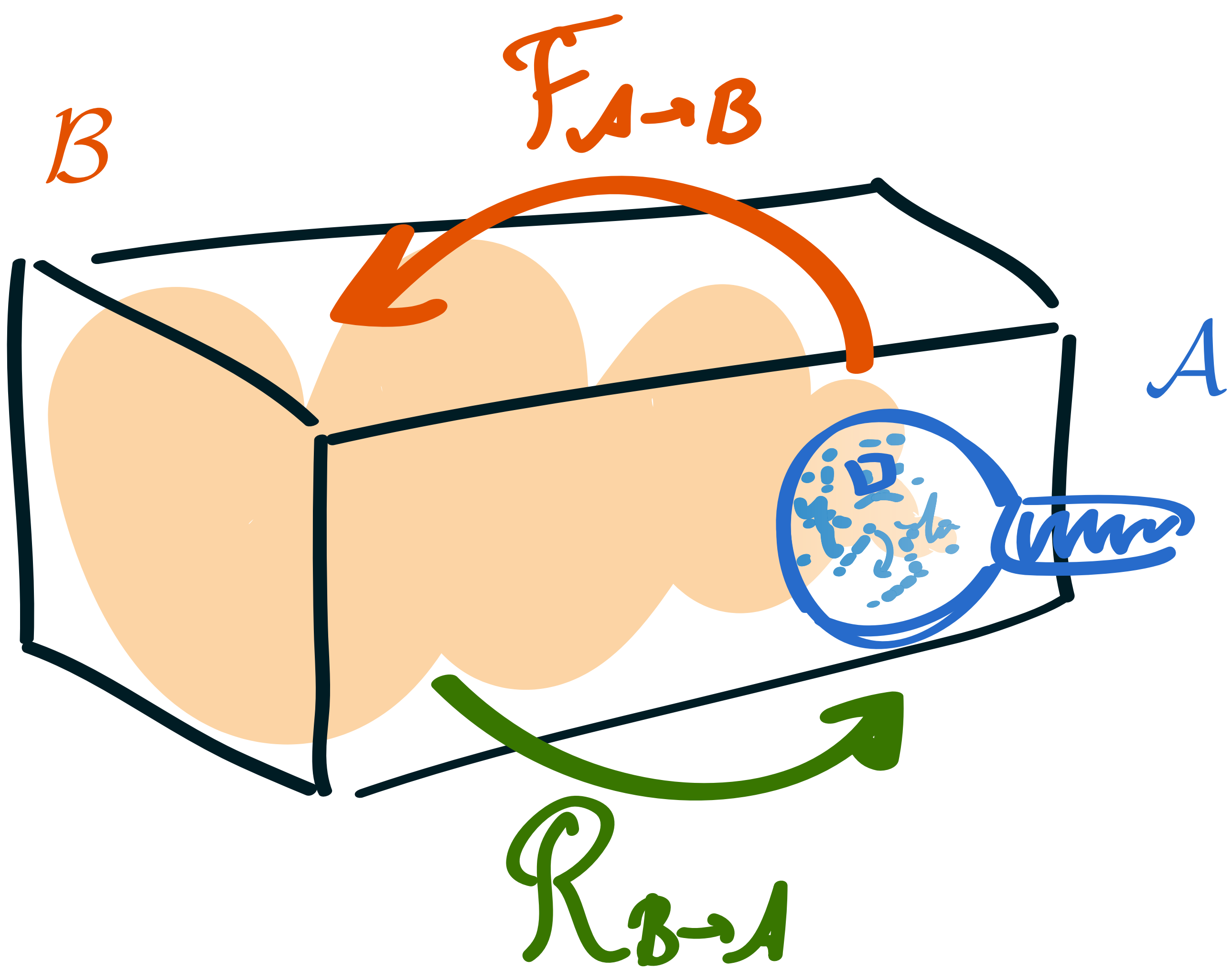


Observers in Thermodynamics



$$\Gamma_B = \mathcal{F}(\Gamma_A)$$

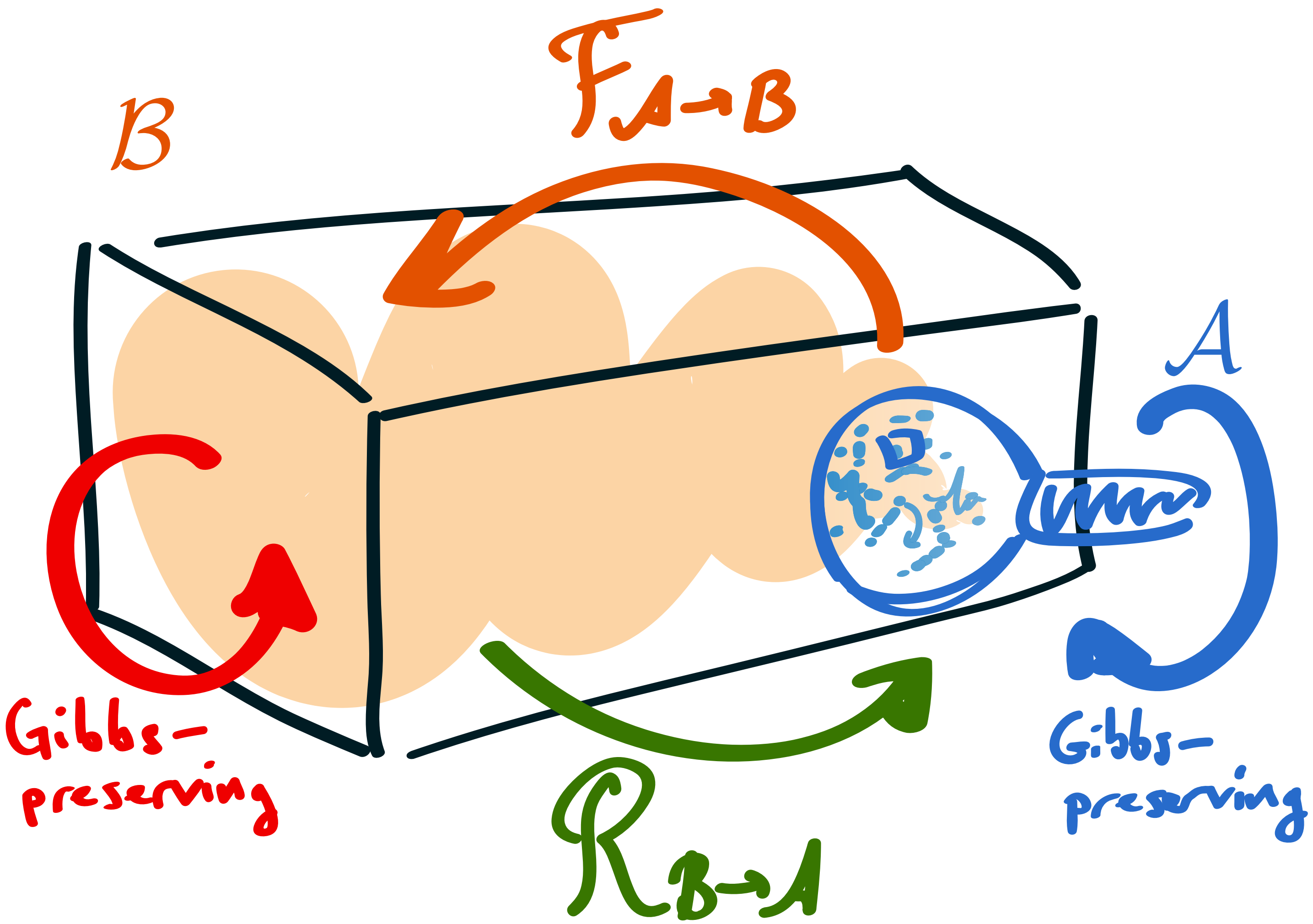
Observers in Thermodynamics



$$\Gamma_B = \mathcal{F}(\Gamma_A)$$
$$\mathcal{R}(\Gamma_B) = \Gamma_A$$

Petz, CMP (1986);
Fawzi & Renner CMP (2015);
Wilde PRSA (2015); ...

Observers in Thermodynamics



$$\Gamma_B = \mathcal{F}(\Gamma_A)$$
$$\mathcal{R}(\Gamma_B) = \Gamma_A$$

$$\mathcal{E}^{\mathcal{B}} = \mathcal{F} \circ \mathcal{E}^{\mathcal{A}} \circ \mathcal{R}$$

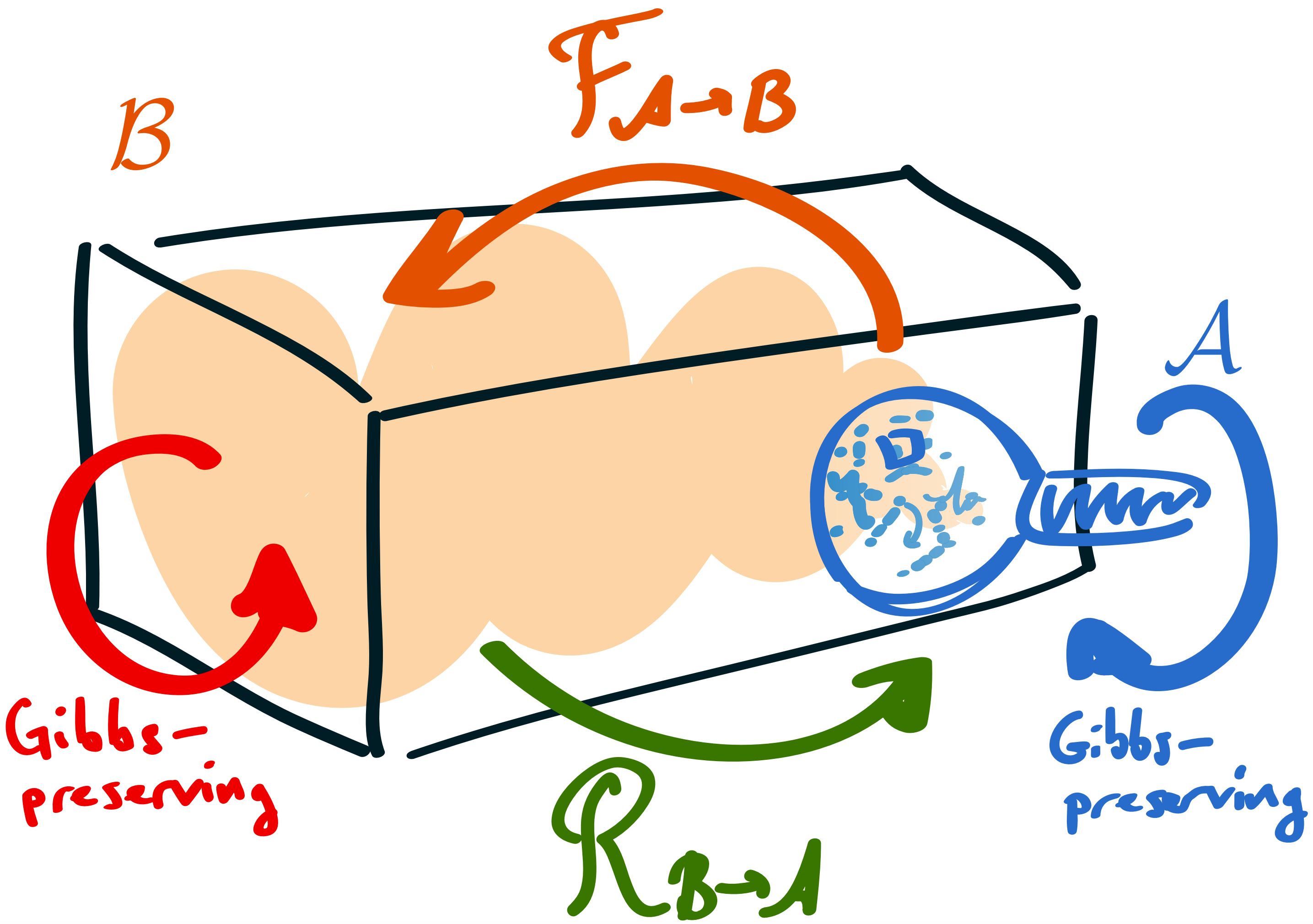
$$\mathcal{E}^{\mathcal{A}}(\Gamma_A) \leqslant \Gamma_A$$

implies

$$\mathcal{E}^{\mathcal{B}}(\Gamma_B) \leqslant \Gamma_B$$

Petz, CMP (1986);
Fawzi & Renner CMP (2015);
Wilde PRSA (2015); ...

Observers in Thermodynamics



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implies

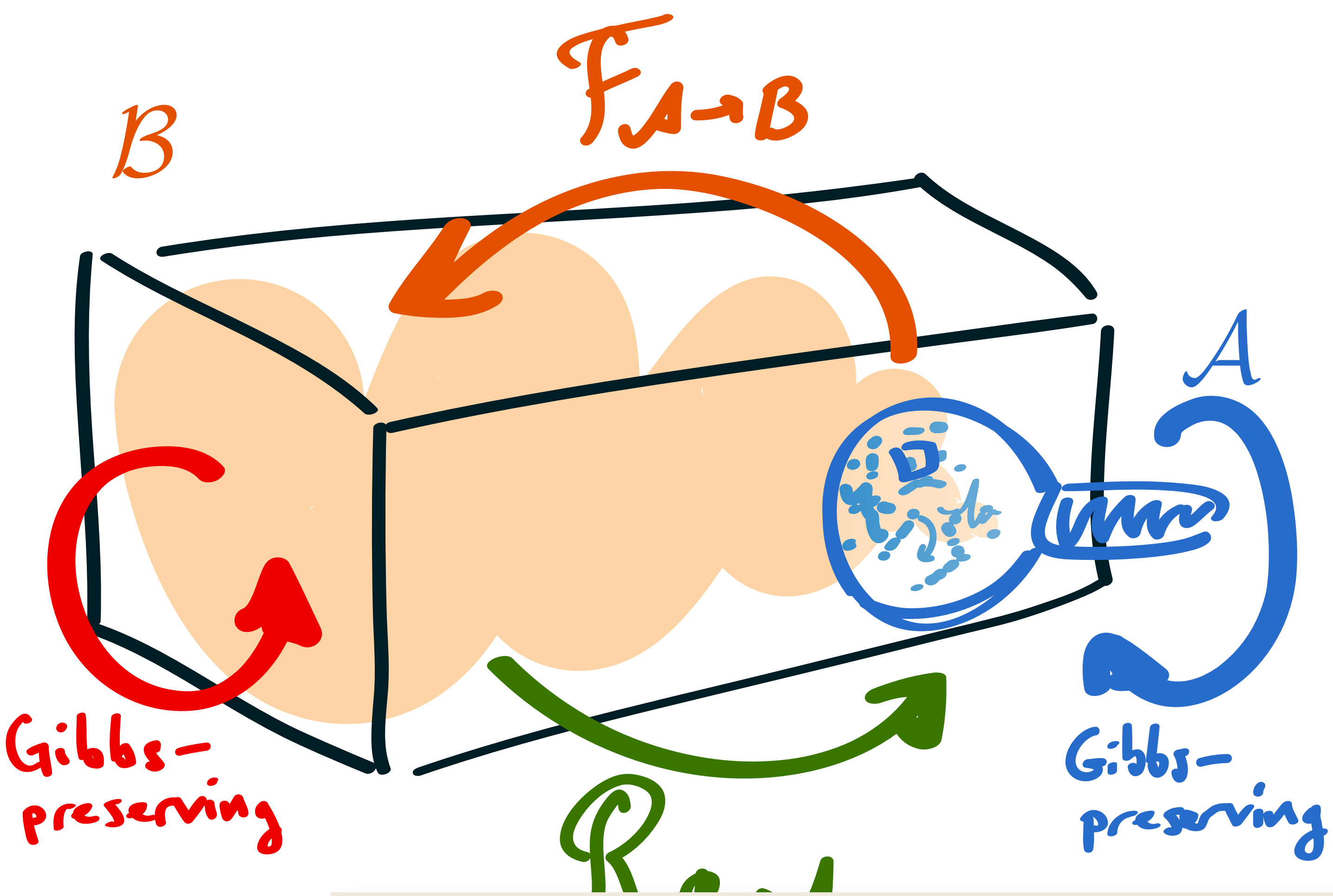
$$\mathcal{E}^{\mathcal{B}}(\Gamma_B) \leqslant \Gamma_B$$

What if $\rho^{\mathcal{A}} \neq \mathcal{R}(\rho^{\mathcal{B}})$?

- **possible apparent violation of second law**

Petz, CMP (1986);
Fawzi & Renner CMP (2015);
Wilde PRSA (2015); ...

Observers in Thermodynamics



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implies

$$\mathcal{E}^{\mathcal{B}}(\Gamma_B) \leqslant \Gamma_B$$

What if ρ

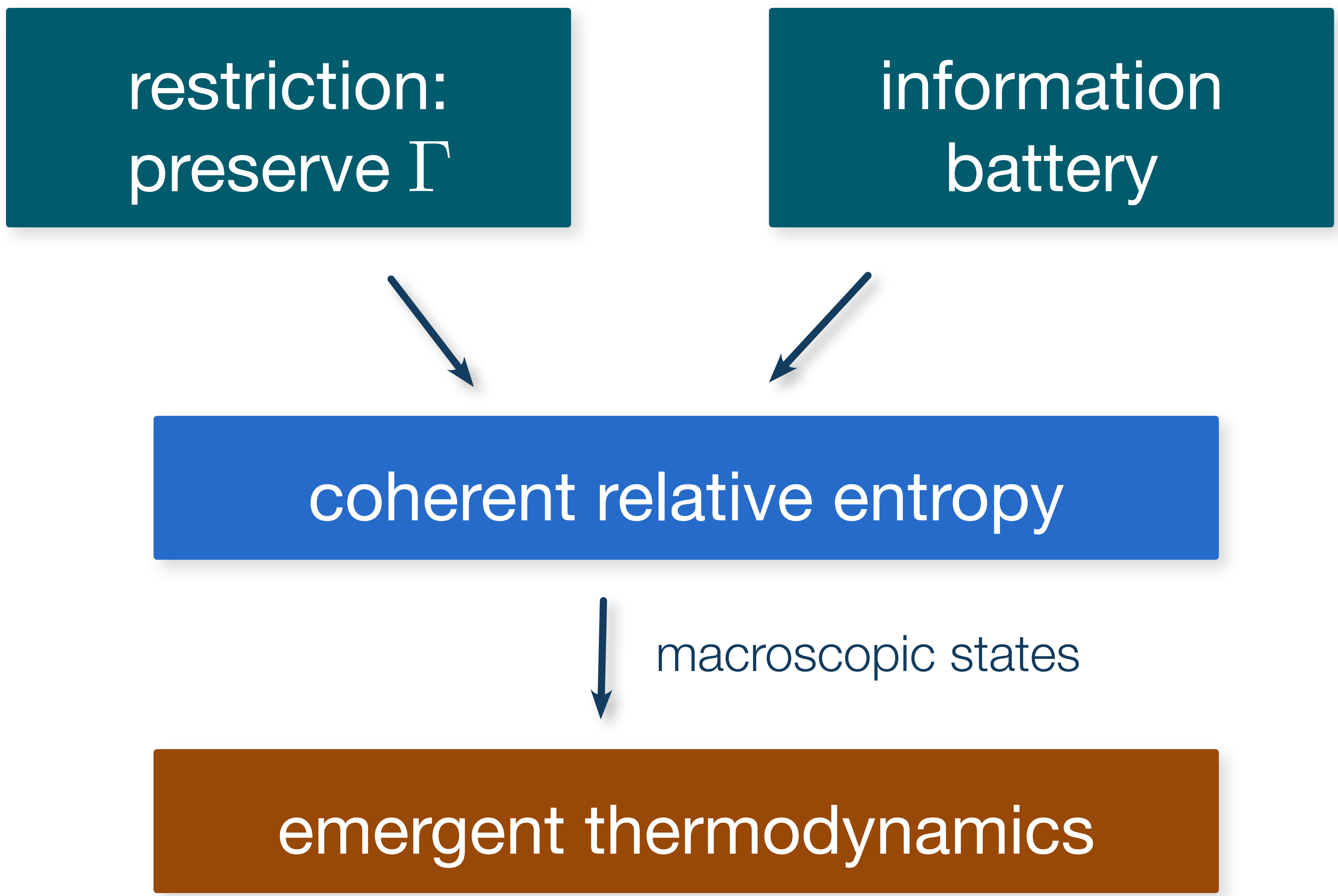
- possi
  secor

- **Criterion for when coarse-grained laws of thermodynamics hold:**

$$\rho^{\mathcal{A}} = \mathcal{R}(\rho^{\mathcal{B}})$$

P (1986);
P (2015);
2015); ...

A picture of thermodynamics



Physics	Information Theory
Hamiltonian	quantum state
time evolution	unitary operation

Physics	Information Theory
Hamiltonian	quantum state
time evolution	unitary operation
energy, number of particles	

Physics

Hamiltonian

time evolution

energy, number of particles

Information Theory

quantum state

unitary operation

thermodynamics (at least 2nd law)

Outlook

- A simple yet general, single-instance, observer-dependent view of thermodynamics
  → better understanding of universality of thermodynamics
- New measure of information
  → non-i.i.d. version of **relative entropy difference**
- Achievability with thermal operations (+ ...)?
- Applications to information theory, coding?
- Applications to physical systems?

Thank you for

your attention!

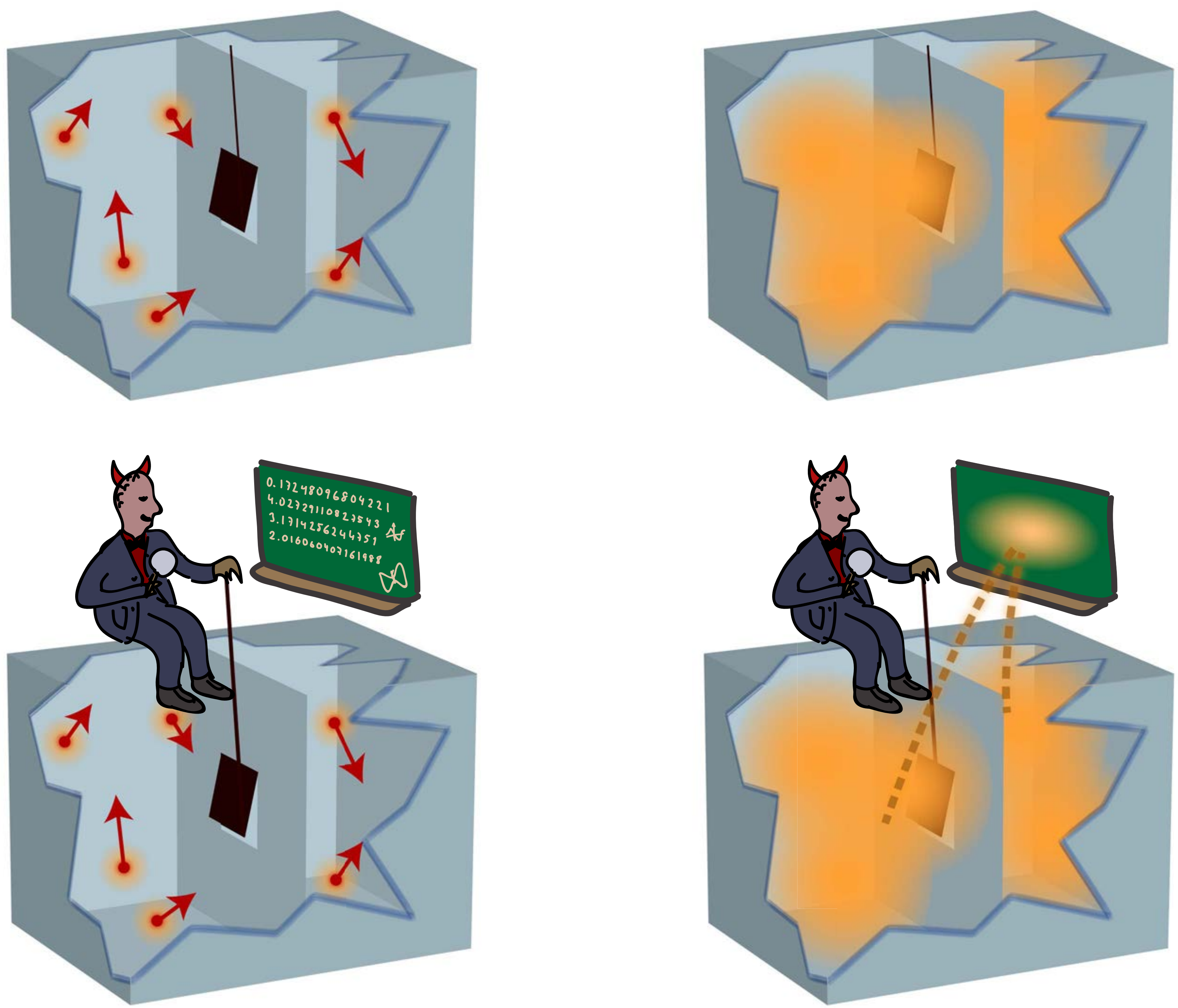
Example: Maxwell's Demon

microscopic picture

macroscopic picture

implicit demon

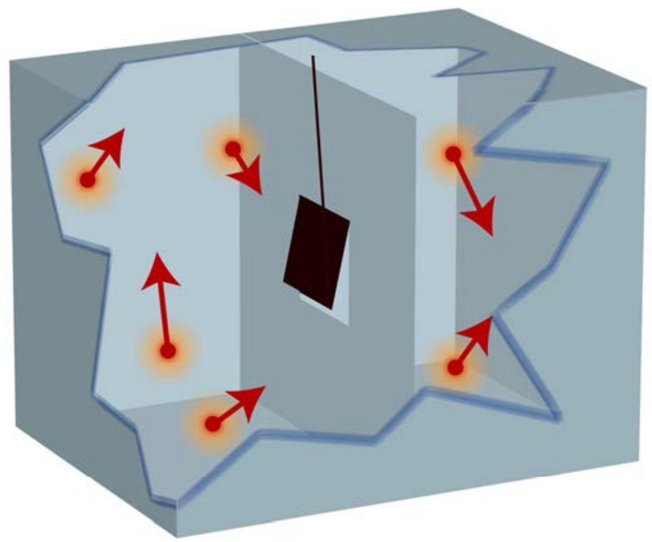
explicit demon

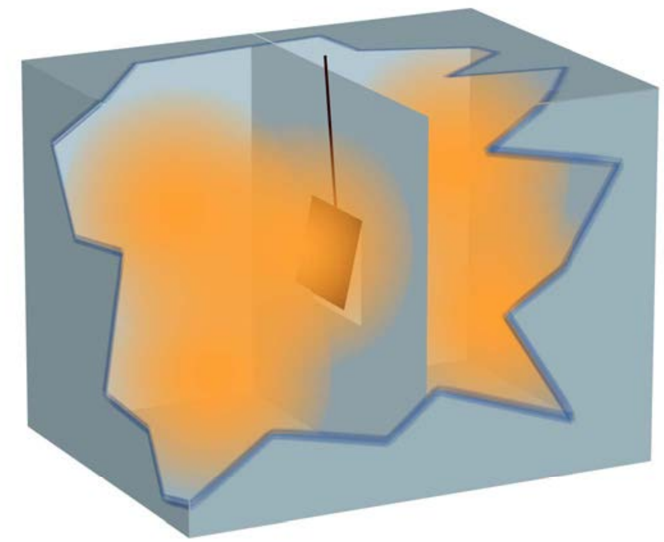


Example: Maxwell's Demon

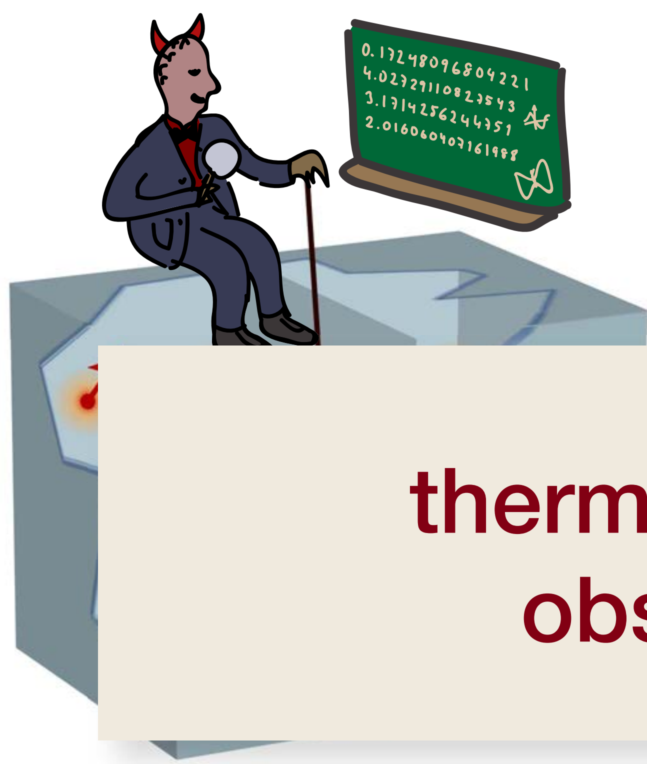
microscopic picture

macroscopic picture





implicit
demon



xplicit
lemon

thermodynamic entropy is observer-dependent!

General case: non-trivial Hamiltonian

Fundamental work cost for any process

$$W = kT \ln 2 \cdot \hat{D}^{\epsilon}_{X \to X'}(\rho_{X'R_X} \,\|\, \Gamma_X, \Gamma_{X'})$$

coherent
relative
entropy

process matrix,
characterizes $\mathcal{E}$
and input state

measures information
relative to Gibbs weights
$\Gamma = e^{-\beta H}$

Approaches to information thermodynamics

statistical mechanics

Piechocinska, PRA, 2000

…

resource theory approach

Brandão *et al.*, PRL, 2013

…

axiomatic approach

Lieb & Yngvason, PR, 1999
Weilenmann *et al.*, PRL, 2016

…

Approaches to information thermodynamics

statistical mechanics

Piechocinska, PRA, 2000

…

work probability distributions, time evolution, fluctuation relations

resource theory approach

Brandão *et al.*, PRL, 2013

…

inherently one-shot, epsilon-work, general processes

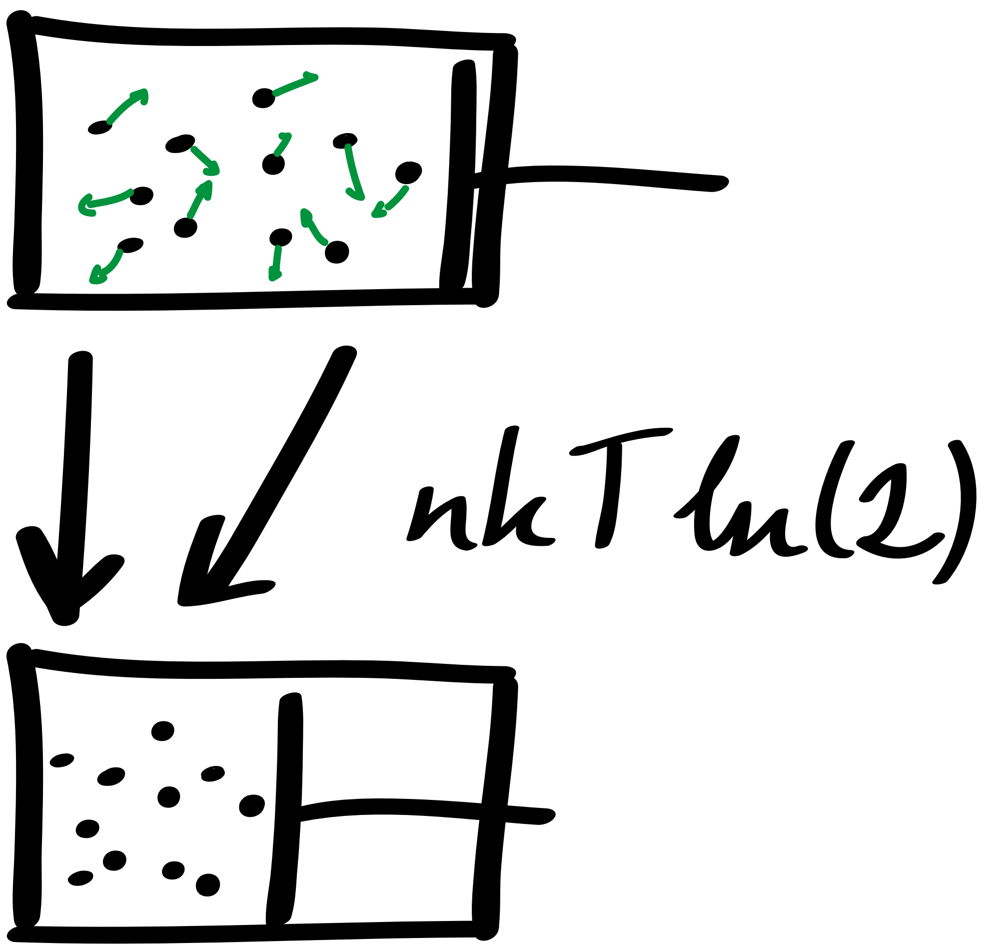
axiomatic approach

Lieb & Yngvason, PR, 1999
Weilenmann *et al.*, PRL, 2016

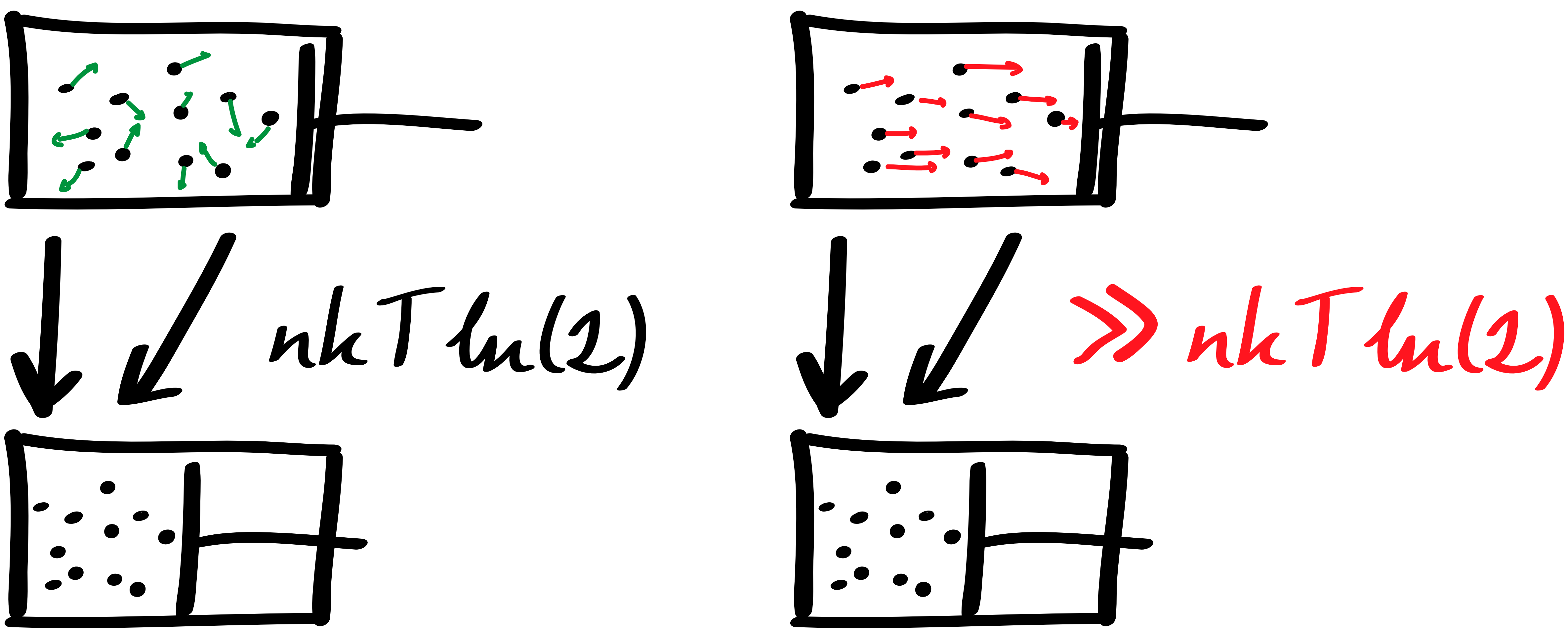
…

abstract, first-principles approach, structure of thermodynamics

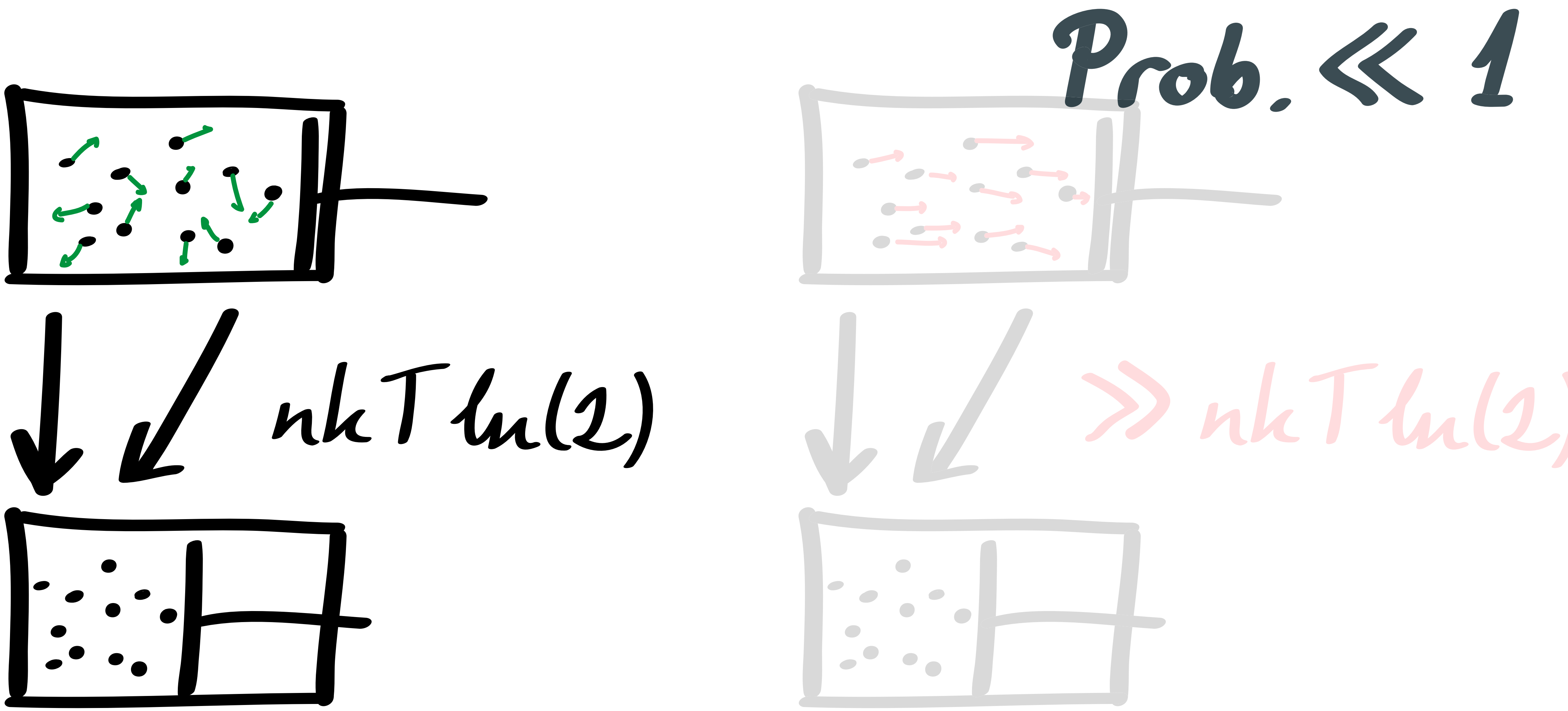
Smoothing



Smoothing



Smoothing



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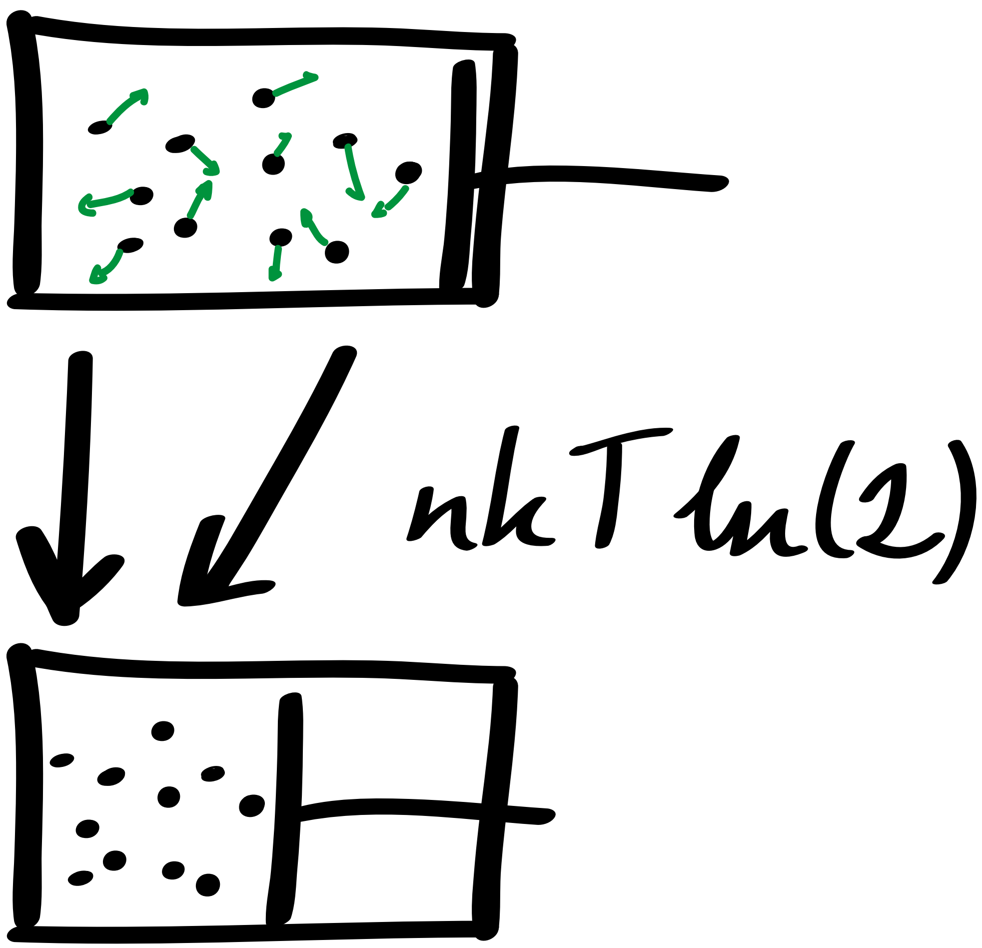
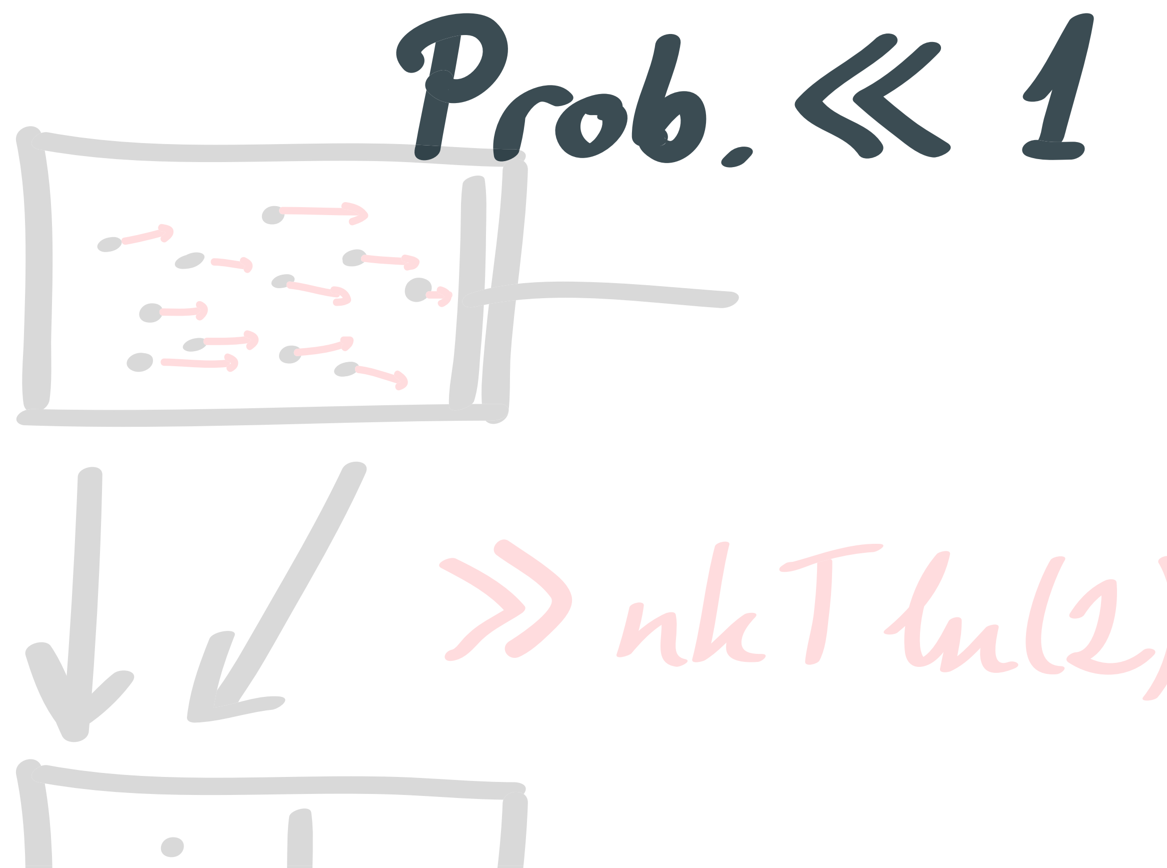




Ignore unlikely events up
to total probability ϵ

What About Statistical Mechanics?

- Model system's time evolution
- Average energy, von Neumann entropy; one-shot statements more tricky
- Count work? p(W) not well defined quantum
- Closer to applications than resource-theory approaches

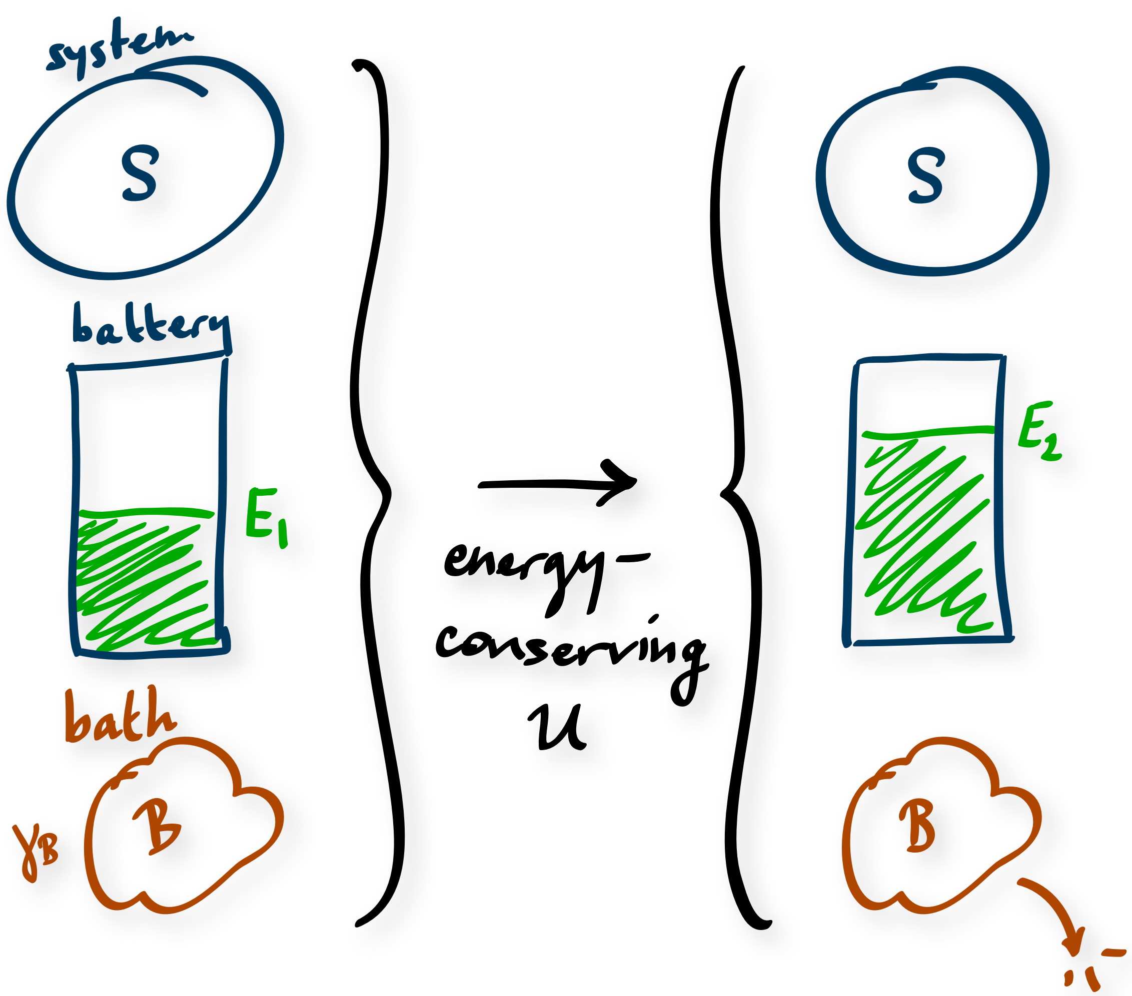
Work?

Count *work* using a battery system

Horodecki & Oppenheim, Nat. Comm. 2013

Work?

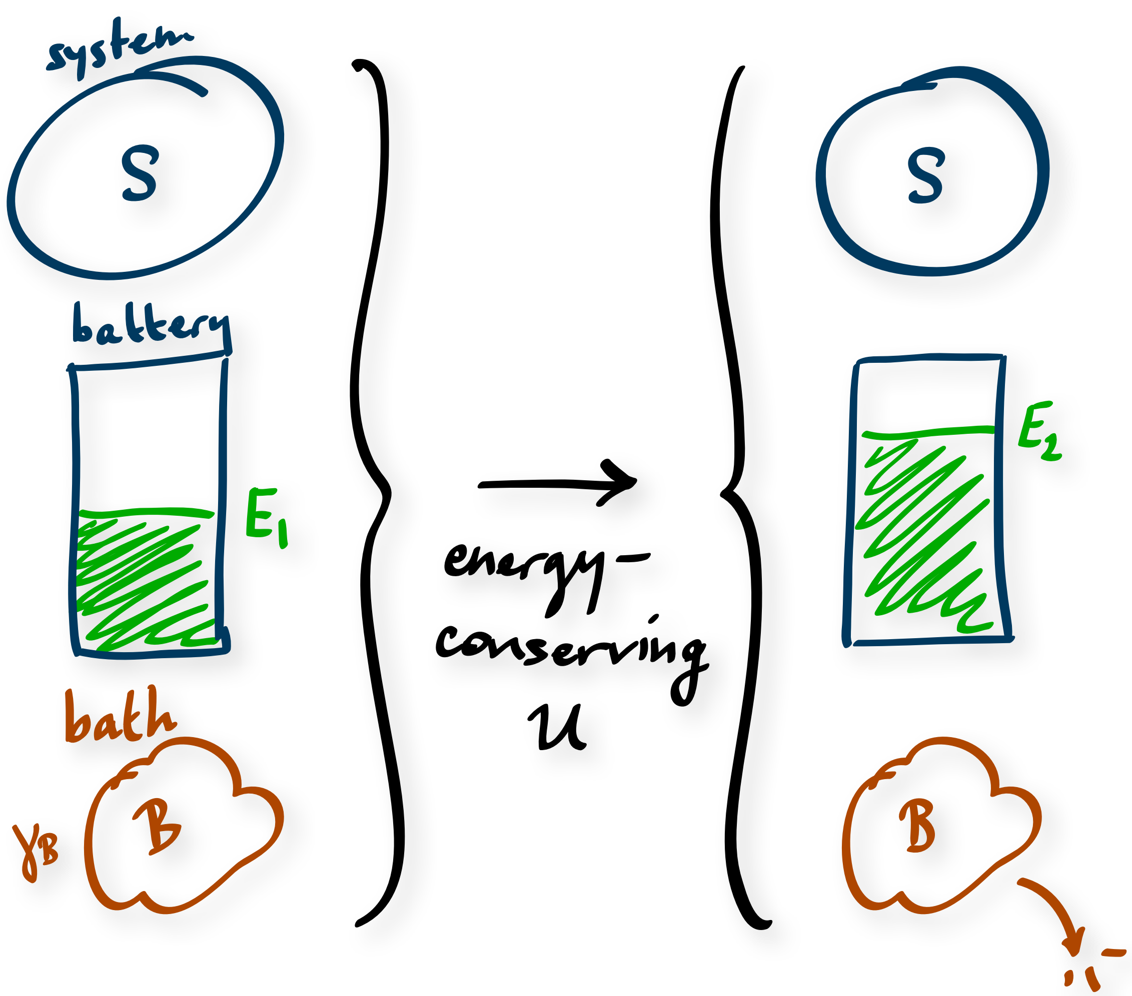
Count *work* using a battery system



Horodecki & Oppenheim, Nat. Comm. 2013

Work?

Count *work* using a battery system



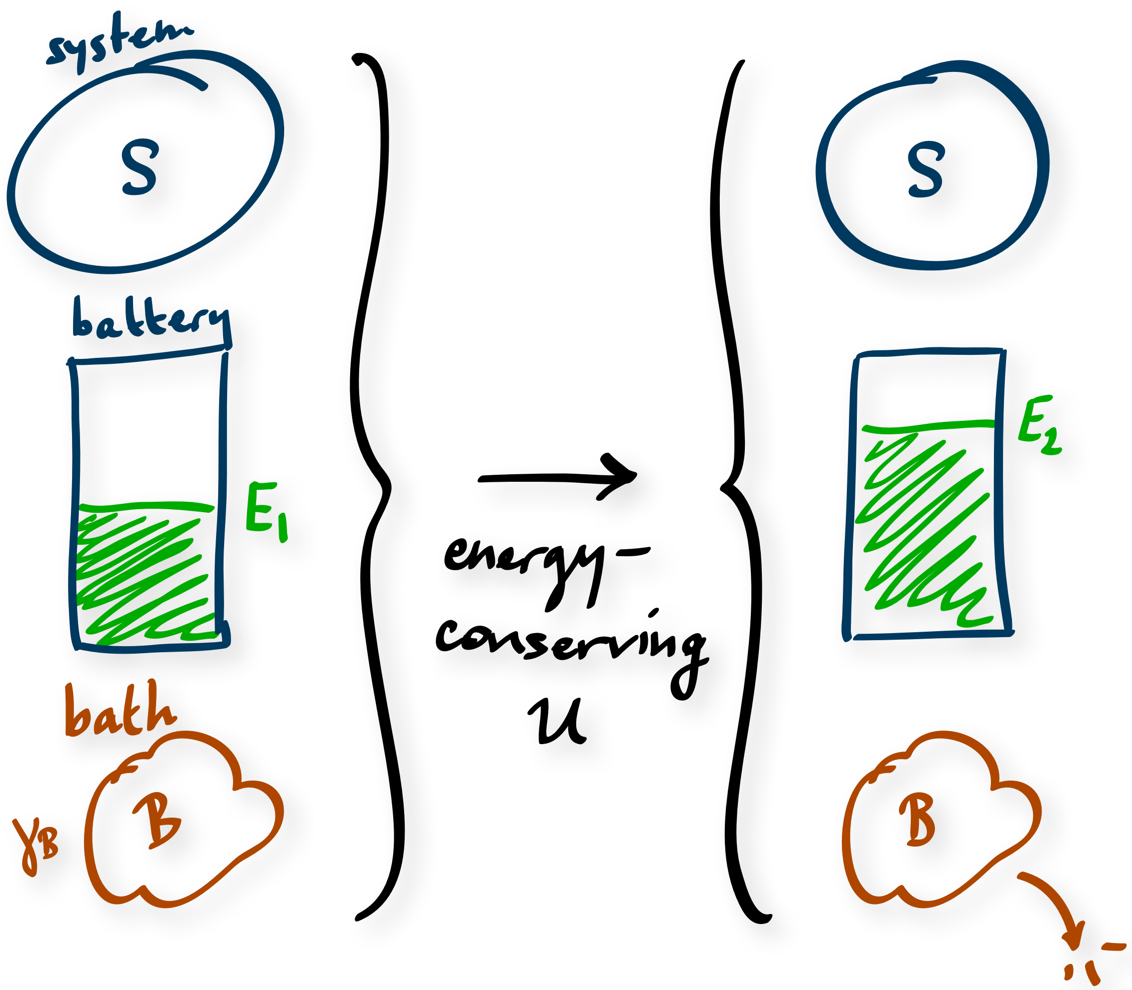
Work extraction $\rho \to \gamma$:

$$E_2 - E_1 = F^{\epsilon}_{\min}(\rho)$$

Horodecki & Oppenheim, Nat. Comm. 2013

Work?

Count *work* using a battery system



Work extraction $\rho \to \gamma$:

$$E_2 - E_1 = F^{\epsilon}_{\min}(\rho)$$

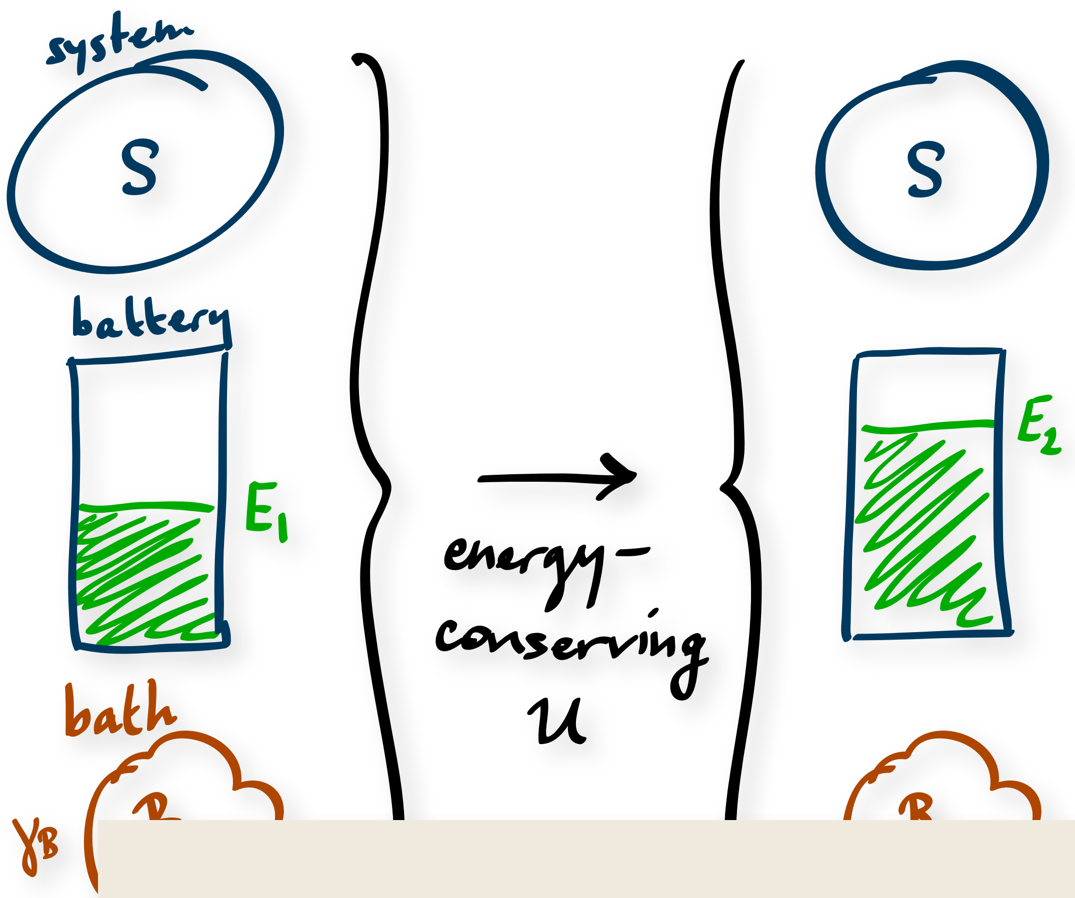
Work cost of formation $\gamma \to \rho$:

$$E_1 - E_2 = F^{\epsilon}_{\max}(\rho)$$

Horodecki & Oppenheim, Nat. Comm. 2013

Work?

Count *work* using a battery system



Work extraction $\rho \to \gamma$:

$$E_2 - E_1 = F^{\epsilon}_{\min}(\rho)$$

Work cost of formation $\gamma \to \rho$:

$$E_1 - E_2 = F^{\epsilon}_{\max}(\rho)$$

- **valid for single instance of the process**
- **macroscopic limit** $F^{\epsilon}_{\min}(\rho), F^{\epsilon}_{\max}(\rho) \to F(\rho)$

2013