The Classification of Clifford Gates over Qubits

Daniel Grier, Luke Schaeffer
MIT

Section 1

Introduction

Classification

Idea

Study universality by finding non-universal gate sets.

Classification

Idea

Study universality by finding non-universal gate sets.

- Each non-universal set has some property (an *invariant*) which is an obstacle to universality.
- Sometimes non-universal sets are interesting in their own right. E.g., Clifford gates.

Classification

Idea

Study universality by finding non-universal gate sets.

- Each non-universal set has some property (an *invariant*) which is an obstacle to universality.
- Sometimes non-universal sets are interesting in their own right. E.g., Clifford gates.

Definition

Let $\langle G \rangle$ denote the set of unitaries constructible as circuits with gates from G.

Classification

Idea

Study universality by finding non-universal gate sets.

- Each non-universal set has some property (an *invariant*) which is an obstacle to universality.
- Sometimes non-universal sets are interesting in their own right. E.g., Clifford gates.

Definition

Let $\langle G \rangle$ denote the set of unitaries constructible as circuits with gates from G.

Goal: Characterize all possible sets $\langle G \rangle$.

Warm up: Reversible Gates

Definition

A gate is *reversible* if it maps each classical input to some classical output.

E.g., CNOT, the Toffoli gate, CSWAP.

Warm up: Reversible Gates

Definition

A gate is *reversible* if it maps each classical input to some classical output.

E.g., CNOT, the Toffoli gate, CSWAP.

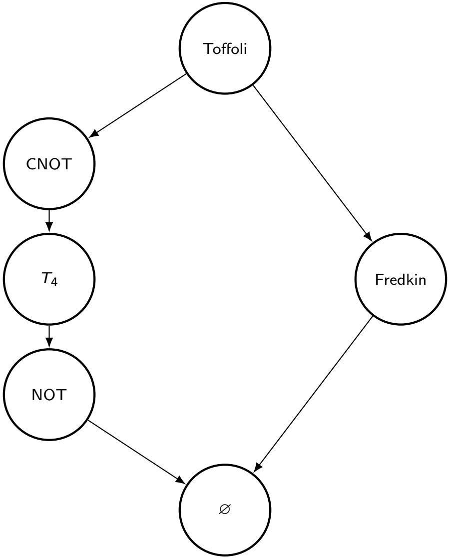
- Completely classified (Aaronson, Grier, S. 2017) in a classical circuit model.
- We will consider a simplified version under a quantum circuit model.

Reversible Gate Classification

Theorem

Given a collection G of reversible gates, $\langle G\rangle$ is one of the following six sets:

- $\langle \mathrm{Toffoli} \rangle$,
- $\langle \mathrm{Fredkin} \rangle$,
- $\langle \mathrm{CNOT} \rangle$,
- $\langle T_4 \rangle$, *(where T_4 is yet to be defined)*
- $\langle \mathrm{NOT} \rangle$, *or*
- $\langle \mathrm{SWAP} \rangle$.


Toffoli
CNOT
T_4
NOT
Fredkin
$\varnothing$

Invariants

Observe that CNOT satisfies the following property.

Affine Invariant

Every output bit is an affine function (modulo 2) of the input bits.

Invariants

Observe that CNOT satisfies the following property.

Affine Invariant

Every output bit is an affine function (modulo 2) of the input bits.

Also, if all gates in a circuit satisfy this property, so does the entire circuit. (This is why it is an *invariant*)

Invariants

Observe that CNOT satisfies the following property.

Affine Invariant

Every output bit is an affine function (modulo 2) of the input bits.

Also, if all gates in a circuit satisfy this property, so does the entire circuit. (This is why it is an *invariant*)

Similarly, Fredkin satisfies the following property.

Conservativity Invariant

The Hamming weight of the input bits is the same the Hamming weight of the output bits.

Once again, if all gates have this property then so does the circuit.

Obstacles to Universality

- If all gates are affine, then the gate set is not universal.
- If all gates are conservative, then the gate set is not universal.

Obstacles to Universality

- If all gates are affine, then the gate set is not universal.
- If all gates are conservative, then the gate set is not universal.

On the other hand, the classification shows us that these are the *only* obstacles.

- If at least one gate is not affine, and at least one gate is not conservative, then the gate set is universal.

Clifford Group

Recall the Pauli matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Pauli group on n qubits is

$$\mathcal{P}_n = \{\pm 1, \pm i\} \times \{I, X, Y, Z\}^{\otimes n}.$$

The Clifford group on n qubits is the set of unitaries that normalize the Pauli group.

$$\mathcal{C}_n = \{U : U\mathcal{P}_n U^\dagger = \mathcal{P}_n\}$$

For example, CNOT $\in \mathcal{C}_2$.

$$\text{CNOT} \cdot (X \otimes I) \cdot \text{CNOT}^\dagger = X \otimes X$$
$$\text{CNOT} \cdot (I \otimes X) \cdot \text{CNOT}^\dagger = I \otimes X$$
$$\text{CNOT} \cdot (Z \otimes I) \cdot \text{CNOT}^\dagger = Z \otimes I$$
$$\text{CNOT} \cdot (I \otimes Z) \cdot \text{CNOT}^\dagger = Z \otimes Z$$

For example, CNOT $\in \mathcal{C}_2$.

$$\begin{aligned}
\mathrm{CNOT} \cdot (X \otimes I) \cdot \mathrm{CNOT}^\dagger &= X \otimes X \\
\mathrm{CNOT} \cdot (I \otimes X) \cdot \mathrm{CNOT}^\dagger &= I \otimes X \\
\mathrm{CNOT} \cdot (Z \otimes I) \cdot \mathrm{CNOT}^\dagger &= Z \otimes I \\
\mathrm{CNOT} \cdot (I \otimes Z) \cdot \mathrm{CNOT}^\dagger &= Z \otimes Z
\end{aligned}$$

Similarly, $H = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right), S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right) \in \mathcal{C}_1$.

$$HXH^\dagger = Z \qquad HZH^\dagger = X$$

$$SXS^\dagger = Y \qquad SZS^\dagger = Z$$

For example, CNOT $\in \mathcal{C}_2$.

$$\text{CNOT} \cdot (X \otimes I) \cdot \text{CNOT}^\dagger = X \otimes X$$
$$\text{CNOT} \cdot (I \otimes X) \cdot \text{CNOT}^\dagger = I \otimes X$$
$$\text{CNOT} \cdot (Z \otimes I) \cdot \text{CNOT}^\dagger = Z \otimes I$$
$$\text{CNOT} \cdot (I \otimes Z) \cdot \text{CNOT}^\dagger = Z \otimes Z$$

Similarly, $H = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right), S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right) \in \mathcal{C}_1$.

$$HXH^\dagger = Z \qquad HZH^\dagger = X$$
$$SXS^\dagger = Y \qquad SZS^\dagger = Z$$

Fact: $\langle \text{CNOT}, H, S \rangle = \bigcup_n \mathcal{C}_n$.

Classification of Clifford Gates

Theorem

Given a collection G of Clifford gates, $\langle G \rangle$ is one of 57 possible classes.

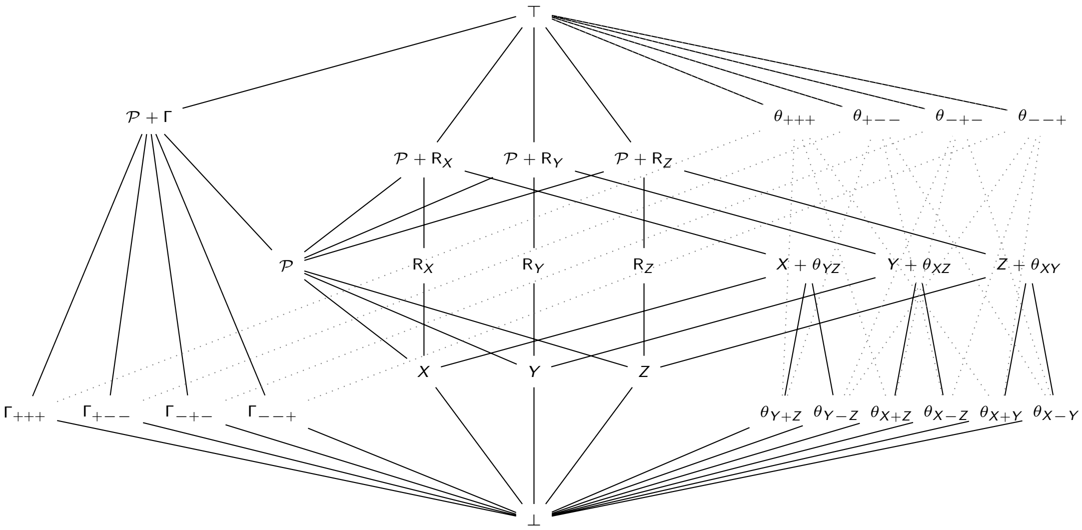
Classification of Clifford Gates

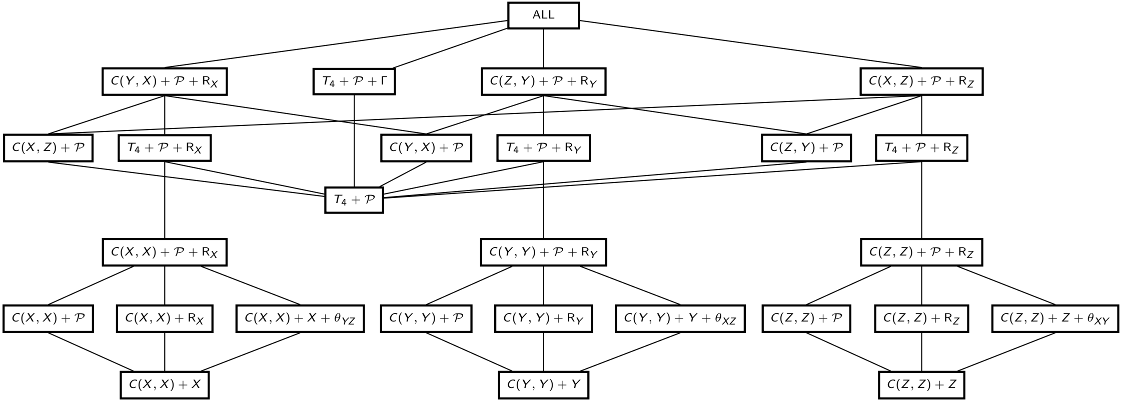
Theorem

Given a collection G of Clifford gates, $\langle G \rangle$ is one of 57 possible classes.

57 sets is too many to list or draw on one slide.

- 30 classes are trivial, generated by single qubit gates.
- 27 more interesting classes


$\top$
$\mathcal{P} + \Gamma$
θ_{+++} θ_{+--} θ_{-+-} θ_{--+}
$\mathcal{P} + \mathrm{R}_X$ $\mathcal{P} + \mathrm{R}_Y$ $\mathcal{P} + \mathrm{R}_Z$
$\mathcal{P}$
R_X R_Y R_Z
$X + \theta_{YZ}$ $Y + \theta_{XZ}$ $Z + \theta_{XY}$
X Y Z
Γ_{+++} Γ_{+--} Γ_{-+-} Γ_{--+}
θ_{Y+Z} θ_{Y-Z} θ_{X+Z} θ_{X-Z} θ_{X+Y} θ_{X-Y}
$\bot$


ALL
$C(Y,X)+\mathcal{P}+\mathrm{R}_X$
$T_4+\mathcal{P}+\Gamma$
$C(Z,Y)+\mathcal{P}+\mathrm{R}_Y$
$C(X,Z)+\mathcal{P}+\mathrm{R}_Z$
$C(X,Z)+\mathcal{P}$
$T_4+\mathcal{P}+\mathrm{R}_X$
$C(Y,X)+\mathcal{P}$
$T_4+\mathcal{P}+\mathrm{R}_Y$
$C(Z,Y)+\mathcal{P}$
$T_4+\mathcal{P}+\mathrm{R}_Z$
$T_4+\mathcal{P}$
$C(X,X)+\mathcal{P}+\mathrm{R}_X$
$C(Y,Y)+\mathcal{P}+\mathrm{R}_Y$
$C(Z,Z)+\mathcal{P}+\mathrm{R}_Z$
$C(X,X)+\mathcal{P}$
$C(X,X)+\mathrm{R}_X$
$C(X,X)+X+\theta_{YZ}$
$C(Y,Y)+\mathcal{P}$
$C(Y,Y)+\mathrm{R}_Y$
$C(Y,Y)+Y+\theta_{XZ}$
$C(Z,Z)+\mathcal{P}$
$C(Z,Z)+\mathrm{R}_Z$
$C(Z,Z)+Z+\theta_{XY}$
$C(X,X)+X$
$C(Y,Y)+Y$
$C(Z,Z)+Z$

Invariants and Tableaux

We state invariants for Clifford gates based on the *tableau* of the gate.

Tableau Intuition

$$\begin{aligned} \text{Clifford gates} &\simeq \text{Linear transformations} \\ \text{Tableaux} &\simeq \text{Matrices} \end{aligned}$$

Invariants and Tableaux

We state invariants for Clifford gates based on the *tableau* of the gate.

Tableau Intuition

$$\begin{aligned} \text{Clifford gates} &\simeq \text{Linear transformations} \\ \text{Tableaux} &\simeq \text{Matrices} \end{aligned}$$

Matrix Representation

- Fix a basis for the vector space.
- Write the image of each basis element as a (linear) combination of basis elements.
- Format data as a table.

Invariants and Tableaux

We state invariants for Clifford gates based on the *tableau* of the gate.

Tableau Intuition

$$\begin{aligned} \text{Clifford gates} &\simeq \text{Linear transformations} \\ \text{Tableaux} &\simeq \text{Matrices} \end{aligned}$$

~~Matrix~~ Tableau Representation

- Fix a basis for the ~~vector space~~ Pauli group.
- Write the image of each basis element as a ~~(linear)~~ combination of basis elements.
- Format data as a table.

Invariants and Tableaux

We state invariants for Clifford gates based on the *tableau* of the gate.

Tableau Intuition

$$\begin{aligned} \text{Clifford gates} &\simeq \text{Linear transformations} \\ \text{Tableaux} &\simeq \text{Matrices} \end{aligned}$$

~~Matrix~~ Tableau Representation

- Fix a basis for the ~~vector space~~ Pauli group.
- Write the image of each basis element as a ~~(linear)~~ combination of basis elements.
- Format data as a table.

Result: $2n \times 2n$ boolean matrix, length $2n$ bit vector (for phase).

Tableau Examples

E.g., CSIGN and H have the following tableaux, ignoring phase bits.

$$\mathrm{CSIGN} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Tableau Examples

E.g., CSIGN and H have the following tableaux, ignoring phase bits.

$$\mathrm{CSIGN} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- $\mathrm{CSIGN}(X \otimes I)\,\mathrm{CSIGN}^\dagger = (X \otimes I)(I \otimes Z)$.
- Basis is $X \otimes I, Z \otimes I, I \otimes X, I \otimes Z$.
- First row is $1, 0, 0, 1$

Invariant Example

$$\mathrm{CSIGN} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Definition

A gate is Z-preserving if there is a 0 in the bottom left of every block of the tableau.

CSIGN is Z-preserving, H is not.

Outline

- Circuit Model
- Quick Tour of the Lattice
  - Important gates
  - Broad structural description
- Elements of the proof
  - Universal construction

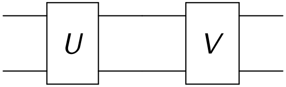
Section 2

Circuit Model

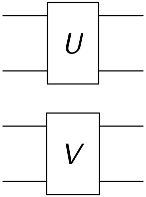
Axiom 1

Given circuits for U and V, we have circuits for UV and $U \otimes V$.

Composition

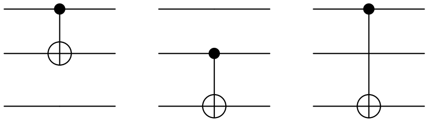


Tensor Product

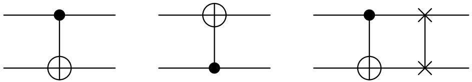


Axiom 2 (Permutation)

1. Does locality matter?

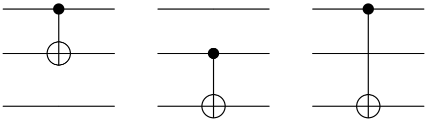


2. Does order matter?

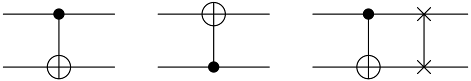


Axiom 2 (Permutation)

1. Does locality matter?



2. Does order matter?



Our Solution

Assume SWAP is in our gate set.

Axiom 3 (Ancillas)

Our ancilla policy allows

1. arbitrarily many ancilla qubits,

Axiom 3 (Ancillas)

Our ancilla policy allows

1. arbitrarily many ancilla qubits,
2. initialized to arbitrarily complex quantum states,

Axiom 3 (Ancillas)

Our ancilla policy allows

1. arbitrarily many ancilla qubits,
2. initialized to arbitrarily complex quantum states,
3. but they must be returned to original state.

Ancilla Qubit Objections

In principle, couldn't the ancilla states be impractical to construct?

Ancilla Qubit Objections

In principle, couldn't the ancilla states be impractical to construct?

- Maybe, but they only need to be constructed once, then reused.
- For this classification we use a discrete set of 1 and 2 qubit ancilla states.

Ancilla Qubit Objections

In principle, couldn't the ancilla states be impractical to construct?

- Maybe, but they only need to be constructed once, then reused.
- For this classification we use a discrete set of 1 and 2 qubit ancilla states.

Clifford gates and magic states are universal for quantum computation, aren't they?

Ancilla Qubit Objections

In principle, couldn't the ancilla states be impractical to construct?

- Maybe, but they only need to be constructed once, then reused.
- For this classification we use a discrete set of 1 and 2 qubit ancilla states.

Clifford gates and magic states are universal for quantum computation, aren't they?

- Yes, but only if the magic states are destroyed in the process, and our ancillas must be returned intact.

Section 3

Tour of the Lattice

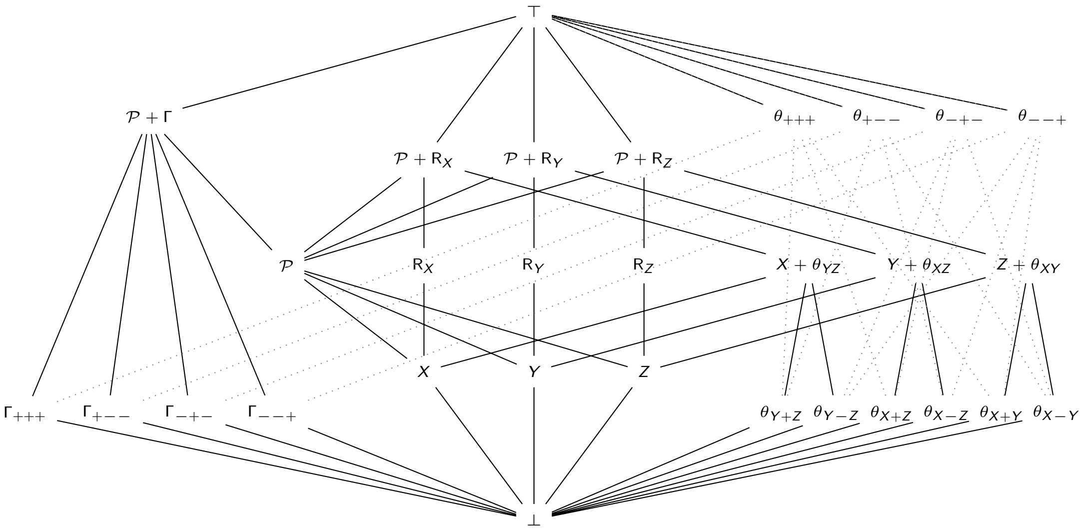
Reminder

Theorem

Given a collection G of Clifford gates, $\langle G \rangle$ is one of 57 possible classes.

The 57 classes break into two major groups:

- 30 classes generated by single qubit gates, and
- 27 other classes.


$\top$
$\mathcal{P} + \Gamma$
θ_{+++}
θ_{+--}
θ_{-+-}
θ_{--+}
$\mathcal{P} + \mathsf{R}_X$
$\mathcal{P} + \mathsf{R}_Y$
$\mathcal{P} + \mathsf{R}_Z$
$\mathcal{P}$
R_X
R_Y
R_Z
$X + \theta_{YZ}$
$Y + \theta_{XZ}$
$Z + \theta_{XY}$
X
Y
Z
Γ_{+++}
Γ_{+--}
Γ_{-+-}
Γ_{--+}
θ_{Y+Z}
θ_{Y-Z}
θ_{X+Z}
θ_{X-Z}
θ_{X+Y}
θ_{X-Y}
$\bot$

Understanding Single Qubit Classes

Fact

Each subgroup of $\mathcal{C}_1$ defines a class.

Each class generated by single qubit gates corresponds to a subgroup of $\mathcal{C}_1$.

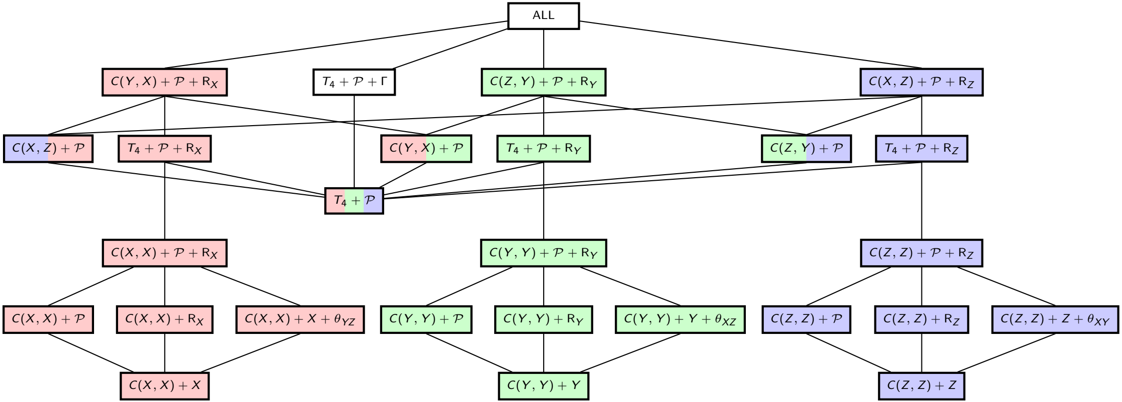
Understanding Single Qubit Classes

Fact

Each subgroup of $\mathcal{C}_1$ defines a class.

Each class generated by single qubit gates corresponds to a subgroup of $\mathcal{C}_1$.

- $\mathcal{C}_1$ is isomorphic to the symmetries of the cube, or S_4.
- These have 30 subgroups, hence 30 classes.


ALL
$C(Y,X)+\mathcal{P}+\mathrm{R}_X$
$T_4+\mathcal{P}+\Gamma$
$C(Z,Y)+\mathcal{P}+\mathrm{R}_Y$
$C(X,Z)+\mathcal{P}+\mathrm{R}_Z$
$C(X,Z)+\mathcal{P}$
$T_4+\mathcal{P}+\mathrm{R}_X$
$C(Y,X)+\mathcal{P}$
$T_4+\mathcal{P}+\mathrm{R}_Y$
$C(Z,Y)+\mathcal{P}$
$T_4+\mathcal{P}+\mathrm{R}_Z$
$T_4+\mathcal{P}$
$C(X,X)+\mathcal{P}+\mathrm{R}_X$
$C(Y,Y)+\mathcal{P}+\mathrm{R}_Y$
$C(Z,Z)+\mathcal{P}+\mathrm{R}_Z$
$C(X,X)+\mathcal{P}$
$C(X,X)+\mathrm{R}_X$
$C(X,X)+X+\theta_{YZ}$
$C(Y,Y)+\mathcal{P}$
$C(Y,Y)+\mathrm{R}_Y$
$C(Y,Y)+Y+\theta_{XZ}$
$C(Z,Z)+\mathcal{P}$
$C(Z,Z)+\mathrm{R}_Z$
$C(Z,Z)+Z+\theta_{XY}$
$C(X,X)+X$
$C(Y,Y)+Y$
$C(Z,Z)+Z$

Color Legend

The colors indicate the gate is X-, Y-, or Z-preserving.

Definition

A gate is Z-preserving if there is a 0 in the bottom left of every block of the tableau.

E.g., CSIGN,

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

Preservation invariants

There are similar invariants for X- and Y-preserving gates. The invariants impose the following restrictions on a typical block $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

- X-preserving: $b = 0$,
- Y-preserving: $a + b + c + d = 0$,
- Z-preserving: $c = 0$.

So CSIGN is not X-preserving or Y-preserving.

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

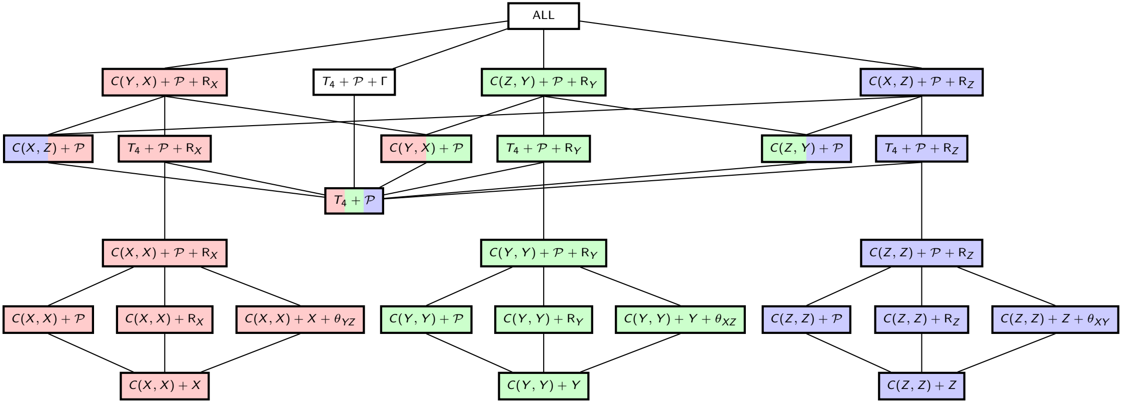
Preservation invariants

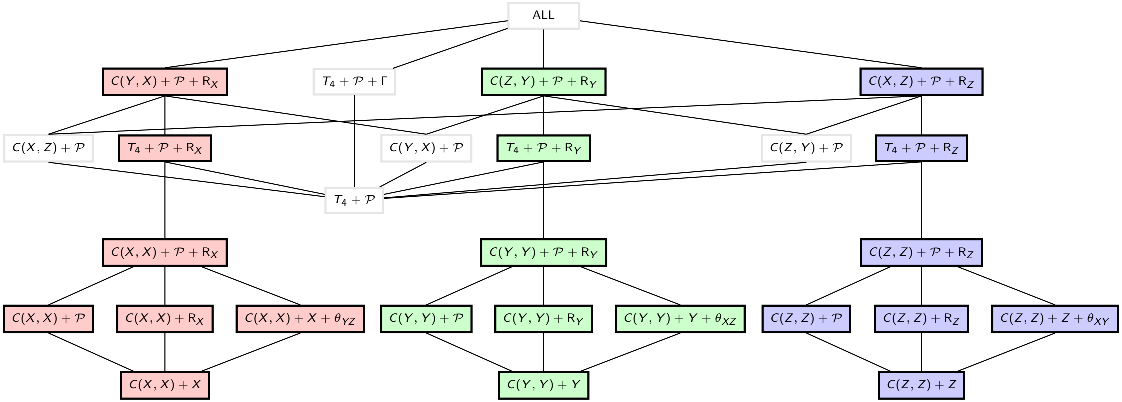
There are similar invariants for X- and Y-preserving gates. The invariants impose the following restrictions on a typical block $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

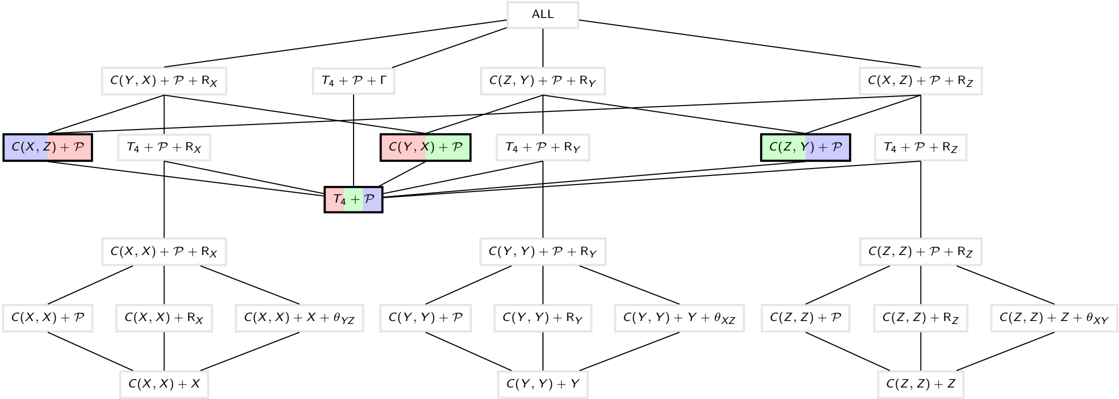
- X-preserving: $b = 0$,
- Y-preserving: $a + b + c + d = 0$,
- Z-preserving: $c = 0$.

But CNOT is X- and Z-preserving!

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$


ALL
$C(Y,X)+\mathcal{P}+\mathrm{R}_X$
$T_4+\mathcal{P}+\Gamma$
$C(Z,Y)+\mathcal{P}+\mathrm{R}_Y$
$C(X,Z)+\mathcal{P}+\mathrm{R}_Z$
$C(X,Z)+\mathcal{P}$
$T_4+\mathcal{P}+\mathrm{R}_X$
$C(Y,X)+\mathcal{P}$
$T_4+\mathcal{P}+\mathrm{R}_Y$
$C(Z,Y)+\mathcal{P}$
$T_4+\mathcal{P}+\mathrm{R}_Z$
$T_4+\mathcal{P}$
$C(X,X)+\mathcal{P}+\mathrm{R}_X$
$C(Y,Y)+\mathcal{P}+\mathrm{R}_Y$
$C(Z,Z)+\mathcal{P}+\mathrm{R}_Z$
$C(X,X)+\mathcal{P}$
$C(X,X)+\mathrm{R}_X$
$C(X,X)+X+\theta_{YZ}$
$C(Y,Y)+\mathcal{P}$
$C(Y,Y)+\mathrm{R}_Y$
$C(Y,Y)+Y+\theta_{XZ}$
$C(Z,Z)+\mathcal{P}$
$C(Z,Z)+\mathrm{R}_Z$
$C(Z,Z)+Z+\theta_{XY}$
$C(X,X)+X$
$C(Y,Y)+Y$
$C(Z,Z)+Z$


ALL
$C(Y,X)+\mathcal{P}+\mathrm{R}_X$
$T_4+\mathcal{P}+\Gamma$
$C(Z,Y)+\mathcal{P}+\mathrm{R}_Y$
$C(X,Z)+\mathcal{P}+\mathrm{R}_Z$
$C(X,Z)+\mathcal{P}$
$T_4+\mathcal{P}+\mathrm{R}_X$
$C(Y,X)+\mathcal{P}$
$T_4+\mathcal{P}+\mathrm{R}_Y$
$C(Z,Y)+\mathcal{P}$
$T_4+\mathcal{P}+\mathrm{R}_Z$
$T_4+\mathcal{P}$
$C(X,X)+\mathcal{P}+\mathrm{R}_X$
$C(Y,Y)+\mathcal{P}+\mathrm{R}_Y$
$C(Z,Z)+\mathcal{P}+\mathrm{R}_Z$
$C(X,X)+\mathcal{P}$
$C(X,X)+\mathrm{R}_X$
$C(X,X)+X+\theta_{YZ}$
$C(Y,Y)+\mathcal{P}$
$C(Y,Y)+\mathrm{R}_Y$
$C(Y,Y)+Y+\theta_{XZ}$
$C(Z,Z)+\mathcal{P}$
$C(Z,Z)+\mathrm{R}_Z$
$C(Z,Z)+Z+\theta_{XY}$
$C(X,X)+X$
$C(Y,Y)+Y$
$C(Z,Z)+Z$


ALL
$C(Y,X)+\mathcal{P}+\mathrm{R}_X$
$T_4+\mathcal{P}+\Gamma$
$C(Z,Y)+\mathcal{P}+\mathrm{R}_Y$
$C(X,Z)+\mathcal{P}+\mathrm{R}_Z$
$C(X,Z)+\mathcal{P}$
$T_4+\mathcal{P}+\mathrm{R}_X$
$C(Y,X)+\mathcal{P}$
$T_4+\mathcal{P}+\mathrm{R}_Y$
$C(Z,Y)+\mathcal{P}$
$T_4+\mathcal{P}+\mathrm{R}_Z$
$T_4+\mathcal{P}$
$C(X,X)+\mathcal{P}+\mathrm{R}_X$
$C(Y,Y)+\mathcal{P}+\mathrm{R}_Y$
$C(Z,Z)+\mathcal{P}+\mathrm{R}_Z$
$C(X,X)+\mathcal{P}$
$C(X,X)+\mathrm{R}_X$
$C(X,X)+X+\theta_{YZ}$
$C(Y,Y)+\mathcal{P}$
$C(Y,Y)+\mathrm{R}_Y$
$C(Y,Y)+Y+\theta_{XZ}$
$C(Z,Z)+\mathcal{P}$
$C(Z,Z)+\mathrm{R}_Z$
$C(Z,Z)+Z+\theta_{XY}$
$C(X,X)+X$
$C(Y,Y)+Y$
$C(Z,Z)+Z$

T_4 gate

Definition

Let T_4 be the 4 qubit gate which

- does nothing to even-parity inputs,
- flips all bits of odd-parity inputs.

E.g. $T_4|0000\rangle = |0000\rangle$, and $T_4|0001\rangle = |1110\rangle$.

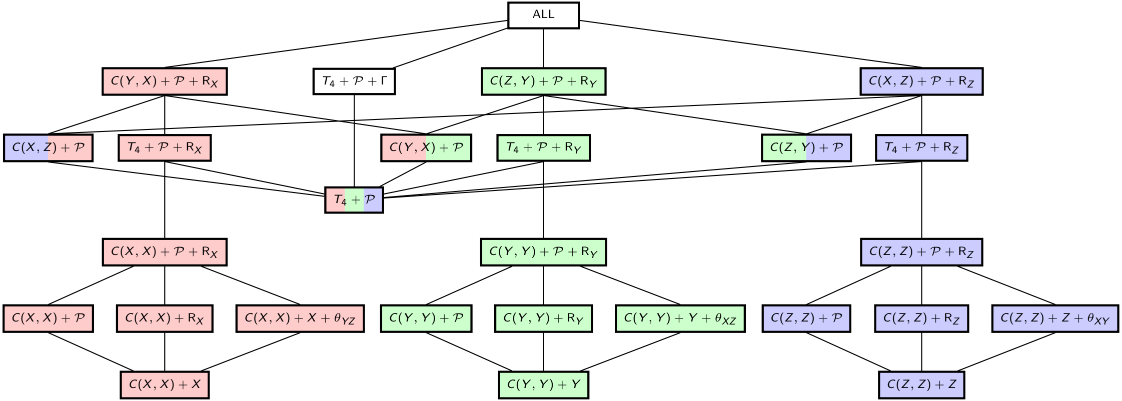
T_4 tableau

The tableau for T_4:

$$\left(\begin{array}{cc|cc|cc|cc}
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
\hline
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0
\end{array}\right)$$

Amazingly, T_4 is X-, Y-, and Z-preserving!

Symmetry?


ALL
$C(Y,X)+\mathcal{P}+R_X$
$T_4+\mathcal{P}+\Gamma$
$C(Z,Y)+\mathcal{P}+R_Y$
$C(X,Z)+\mathcal{P}+R_Z$
$C(X,Z)+\mathcal{P}$
$T_4+\mathcal{P}+R_X$
$C(Y,X)+\mathcal{P}$
$T_4+\mathcal{P}+R_Y$
$C(Z,Y)+\mathcal{P}$
$T_4+\mathcal{P}+R_Z$
$T_4+\mathcal{P}$
$C(X,X)+\mathcal{P}+R_X$
$C(Y,Y)+\mathcal{P}+R_Y$
$C(Z,Z)+\mathcal{P}+R_Z$
$C(X,X)+\mathcal{P}$
$C(X,X)+R_X$
$C(X,X)+X+\theta_{YZ}$
$C(Y,Y)+\mathcal{P}$
$C(Y,Y)+R_Y$
$C(Y,Y)+Y+\theta_{XZ}$
$C(Z,Z)+\mathcal{P}$
$C(Z,Z)+R_Z$
$C(Z,Z)+Z+\theta_{XY}$
$C(X,X)+X$
$C(Y,Y)+Y$
$C(Z,Z)+Z$

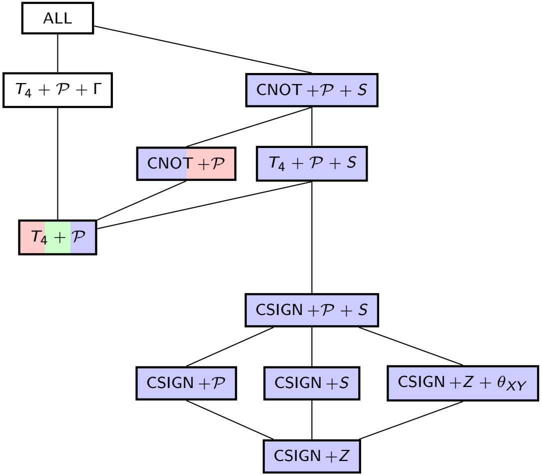
Three-fold symmetry

The automorphism of the Pauli group that maps

$$\begin{aligned} I &\to I \\ X &\to Y \\ Y &\to Z \\ Z &\to X, \end{aligned}$$

lifts to an automorphism of the Clifford group, and an automorphism of the lattice.

Z-preserving and misc. classes



Section 4

Proof Techniques

Proof Techniques

Steps of the proof:

Proof Techniques

Steps of the proof:

1. Characterize each class by generators and an invariant.

   E.g., T_4 generates all gates that are simultaneously X-, Y-, and Z-preserving.

Proof Techniques

Steps of the proof:

1. Characterize each class by generators and an invariant.

   E.g., T_4 generates all gates that are simultaneously X-, Y-, and Z-preserving.

Proof Techniques

Steps of the proof:

1. Characterize each class by generators and an invariant.

   E.g., T_4 generates all gates that are simultaneously X-, Y-, and Z-preserving.

2. Argue that the classes are distinct.

   E.g., CNOT is not in the class generated by T_4 because CNOT is not Y-preserving.

Proof Techniques

Steps of the proof:

1. Characterize each class by generators and an invariant.

   E.g., T_4 generates all gates that are simultaneously X-, Y-, and Z-preserving.

2. Argue that the classes are distinct.

   E.g., CNOT is not in the class generated by T_4 because CNOT is not Y-preserving.

Proof Techniques

Steps of the proof:

1. Characterize each class by generators and an invariant.

   E.g., T_4 generates all gates that are simultaneously X-, Y-, and Z-preserving.

2. Argue that the classes are distinct.

   E.g., CNOT is not in the class generated by T_4 because CNOT is not Y-preserving.

3. Argue that no class has been omitted.

No missing classes

1. Suppose for a contradiction that $\langle G \rangle$ is not a listed class.

No missing classes

1. Suppose for a contradiction that $\langle G \rangle$ is not a listed class.
2. For each invariant G does not satisfy, construct a simple gate violating that invariant.
    - E.g., if some gate is not Y-preserving, extract one of
        - a single qubit gate that is not Y-preserving,
        - a CSIGN,
        - a CNOT, etc.

No missing classes

1. Suppose for a contradiction that $\langle G \rangle$ is not a listed class.
2. For each invariant G does not satisfy, construct a simple gate violating that invariant.
   - E.g., if some gate is not Y-preserving, extract one of
     - a single qubit gate that is not Y-preserving,
     - a CSIGN,
     - a CNOT, etc.
3. Use simple gates to construct canonical class generators.
   - E.g., the class is defined as generated by T_4 and S, but you have T_4 and CSIGN.

1. Identify 2×2 block of the tableau violating the invariant.

$$\begin{pmatrix} & \vdots & \\ \cdots & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \cdots \\ & \vdots & \end{pmatrix}$$

2. Apply a SWAP to make that row the first one.

$$\begin{pmatrix} \cdots & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \cdots \\ & \vdots & \\ & & \end{pmatrix}$$

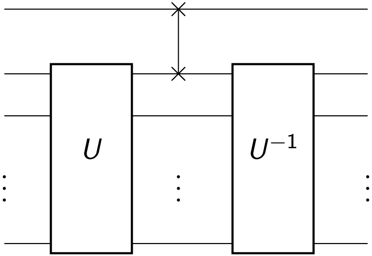
Assume the first row is

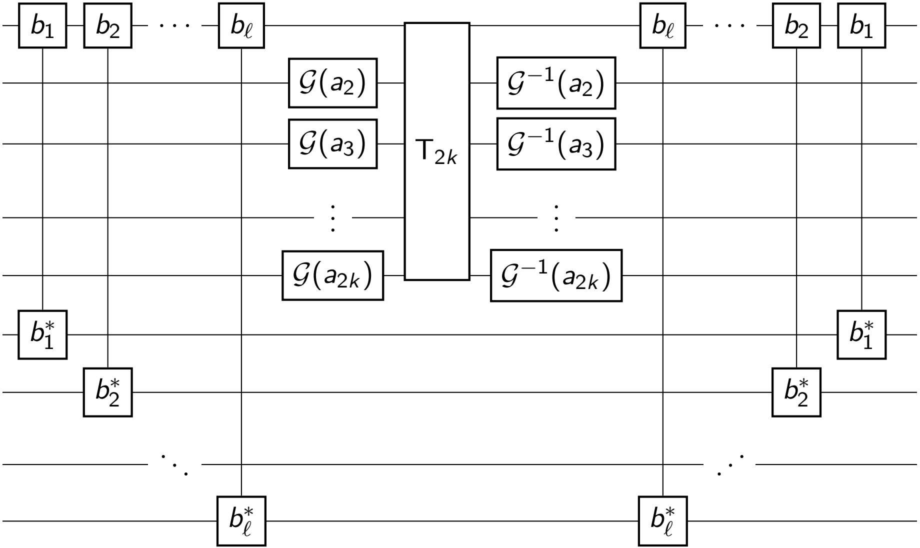
$$\begin{pmatrix} a_1 & a_2 & \dots & a_{2k} & b_1 & \dots & b_\ell & 0 & \dots & 0 \end{pmatrix}$$

where

- $a_1, \dots, a_{2k}$ are invertible 2×2 blocks,
- $b_1, \dots, b_\ell \neq 0$ are non-invertible 2×2 blocks.

Universal Construction




b_1
b_2
b_ℓ
$\mathcal{G}(a_2)$
$\mathcal{G}(a_3)$
$\mathcal{G}(a_{2k})$
T_{2k}
$\mathcal{G}^{-1}(a_2)$
$\mathcal{G}^{-1}(a_3)$
$\mathcal{G}^{-1}(a_{2k})$
b_ℓ
b_2
b_1
b_1^*
b_2^*
b_ℓ^*
b_ℓ^*
b_2^*
b_1^*

Section 5

Open Problems

- Complete the classification of quantum gates. Possible strategy: divide into cases by group of single qubit gates.

- Complete the classification of quantum gates. Possible strategy: divide into cases by group of single qubit gates.
- Classify Clifford gates but with stabilizer state ancillas. We can show nothing changes unless

$$\langle \Gamma \otimes \Gamma^{-1} \rangle \neq \langle \Gamma \rangle,$$

where $\Gamma = HS$.