Universal points in the asymptotic spectrum of tensors

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Asymptotic transformations between tensors

$$s \gtrsim t$$

The asymptotic spectrum of tensors

$$\Delta(\{\text{tensors}\})$$

Quantum functionals

$$F_\theta \in \Delta(\{\text{tensors}\})$$

Tensors

$$t = \left(t_{i_1 i_2 i_3}\right)_{i_1 i_2 i_3} \qquad \in \mathbb{C}^{n_1 \times n_2 \times n_3} \cong \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$$

$$= \sum_{i_1 i_2 i_3} t_{i_1 i_2 i_3} \, |i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle$$

Diagonal tensor

$$\langle n \rangle = (\delta(i_1 = i_2 = i_3))_{i_1 i_2 i_3} \qquad \in \mathbb{C}^{n \times n \times n} \cong \mathbb{C}^{n} \otimes \mathbb{C}^{n} \otimes \mathbb{C}^{n}$$

$$= \sum_{i \in [n]} |i\rangle \otimes |i\rangle \otimes |i\rangle$$

$\langle 2 \rangle$ = Greenberger–Horne–Zeilinger (GHZ) state

Restriction

$t \in \mathbb{C}^{n_1 \times n_2 \times n_3}$
$s \in \mathbb{C}^{m_1 \times m_2 \times m_3}$

$t \geq s$ if there are matrices

$A_1 \in \mathbb{C}^{m_1 \times n_1}$
$A_2 \in \mathbb{C}^{m_2 \times n_2}$
$A_3 \in \mathbb{C}^{m_3 \times n_3}$

such that

$$(A_1 \otimes A_2 \otimes A_3) \cdot t = s$$

$$(I \otimes A_2 \otimes I) \cdot |i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle$$
$$= |i_1\rangle \otimes A_2|i_2\rangle \otimes |i_3\rangle$$

stochastic local operations and classical communication (slocc)

[Dür–Vidal–Cirac‘00]

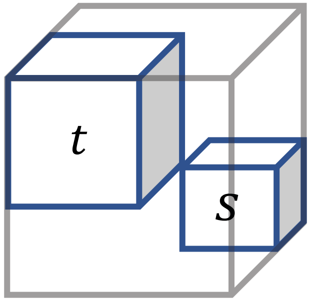
Direct sum and tensor product

$t \in \mathbb{C}^{n_1 \times n_2 \times n_3}$
$s \in \mathbb{C}^{m_1 \times m_2 \times m_3}$

$t \oplus s \qquad \in \mathbb{C}^{(n_1+m_1) \times (n_2+m_2) \times (n_3+m_3)}$

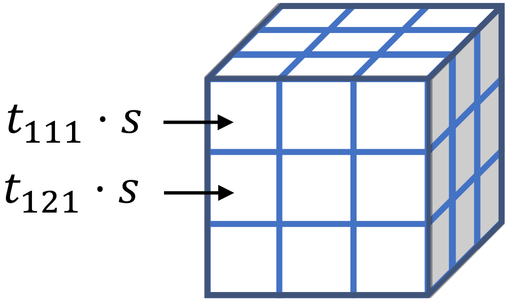
$$\sum_{i_1 i_2 i_3} t_{i_1 i_2 i_3} \, |i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle \ +$$

$$\sum_{j_1 j_2 j_3} s_{j_1 j_2 j_3} \, |n_1 + j_1\rangle \otimes |n_2 + j_2\rangle \otimes |n_3 + j_3\rangle$$



$t \otimes s \qquad \in \mathbb{C}^{(n_1 \cdot m_1) \times (n_2 \cdot m_2) \times (n_3 \cdot m_3)}$

$$\sum_{\substack{i_1 i_2 i_3 \\ j_1 j_2 j_3}} t_{i_1 i_2 i_3} \cdot s_{j_1 j_2 j_3} \, |i_1, j_1\rangle \otimes |i_2, j_2\rangle \otimes |i_3, j_3\rangle$$



Asymptotic restriction

$t \in \mathbb{C}^{n_1 \times n_2 \times n_3}$
$s \in \mathbb{C}^{m_1 \times m_2 \times m_3}$

$t \gtrsim s$ if there are numbers

$\epsilon_N \in \mathbb{N}$ with $\frac{\epsilon_N}{N} \to 0$ when $N \to \infty$

and

$\forall N \in \mathbb{N} \quad t^{\otimes N + \epsilon_N} \geq s^{\otimes N}$

asymptotic slocc transformation at rate 1

Rank and sub-rank

$\langle m \rangle \leq t \leq \langle n \rangle$?

currency	diagonals tensors	$\langle n \rangle \in \mathbb{C}^{n \times n \times n}$
cost of t	rank	$\mathrm{R}(t) = \min\{n \in \mathbb{N} : t \leq \langle n \rangle\} \quad < \infty$
value of t	sub-rank	$\mathrm{Q}(t) = \max\{m \in \mathbb{N} : \langle m \rangle \leq t\} \quad \geq 0$

Computational complexity [Håstad90, Shitov16]
Deciding $\mathrm{R}(t) \leq x$ is NP-hard

Asymptotic rank and asymptotic sub-rank

asymptotic rank $\underset{\sim}{\mathrm{R}}(t) = \lim_{N\to\infty} \mathrm{R}\left(t^{\otimes N}\right)^{1/N}$

asymptotic sub-rank $\underset{\sim}{\mathrm{Q}}(t) = \lim_{N\to\infty} \mathrm{Q}\left(t^{\otimes N}\right)^{1/N}$

$\mathrm{Q}(t) \leq \underset{\sim}{\mathrm{Q}}(t) \leq \underset{\sim}{\mathrm{R}}(t) \leq \mathrm{R}(t)$

Connections

$\underset{\smile}{\mathrm{R}}(\text{“matrix multiplication tensor”}) = 2^{\omega}$ $\qquad 2 \leq \omega \leq 2.3728639\ldots$

- complexity of matrix multiplication in **algebraic complexity theory** [Strassen, Coppersmith, Winograd, …]
- upper bounds nondeterministic multiparty quantum **communication complexity** for pairwise equality in a triangle [Buhrman et al. ‘16]

$\underset{\smile}{\mathrm{Q}}(\text{special tensor})$

- upper bound query complexity in **property testing** as in e.g. [Green‘05, Fu–Kleinberg‘13, …]
- upper bound size of “cap sets” in **combinatorics** (requires finite field) [Ellenberg–Gijswijt‘17, Tao, Tao–Sawin, Blasiak et al. ‘17, Christandl–Vrana–Zuiddam‘17]

$\gtrsim$

- asymptotic slocc transformation in **quantum information theory** [Yu et al. ‘10, Christandl–Vrana‘13, Christandl–Vrana–Zuiddam‘16, …]

Problem: deciding asymptotic restriction

$t \gtrsim s$?

i.e. $t^{\otimes N + o(N)} \geq s^{\otimes N}$?

(or $\underset{\sim}{\mathrm{R}}(t) = ?$ or $\underset{\sim}{\mathrm{Q}}(t) = ?$)

Two directions

- **Constructions**: matrices carrying out $t \gtrsim s$
- **Obstructions**: “certificates” that prohibit $t \gtrsim s$

The asymptotic spectrum of tensors (Strassen 1986)

$\Delta(\{\text{tensors}\}) =$ set of maps $F : \{\text{tensors}\} \to \mathbb{R}_{\geq 0}$ such that

1. if $t \geq s$ then $F(t) \geq F(s)$ — monotone
2. $F(s \oplus t) = F(s) + F(t)$ — additive
3. $F(s \otimes t) = F(s)F(t)$ — multiplicative
4. $F(\langle n \rangle) = n$ for $n \in \mathbb{N}$ — normalized

$$t \gtrsim s \quad \Rightarrow \quad t^{\otimes N + \epsilon_N} \geq s^{\otimes N} \quad \Rightarrow \quad F(t)^{N+\epsilon_N} \geq F(s)^N \quad \Rightarrow \quad F(t) \geq F(s)$$

$F(t) < F(s) \quad \Rightarrow \quad t \not\gtrsim s$ $\quad$ F serves as an obstruction!

$$\underset{\sim}{\mathrm{Q}}(t) \leq F(t) \leq \underset{\sim}{\mathrm{R}}(t)$$

The spectral theorem (Strassen 1986)

$\Delta(\{\text{tensors}\}) = \{ F : \{\text{tensors}\} \to \mathbb{R}_{\geq 0} \;:\; \text{monot}, \text{add}, \text{mult}, \text{norm}\}$

i. $t \gtrsim s$ iff $\forall F \in \Delta(\{\text{tensors}\}) \quad F(t) \geq F(s)$

ii. $\underset{\sim}{\mathrm{Q}}(t) = \min_{F \in \Delta(\{\text{tensors}\})} F(t)$

$\underset{\sim}{\mathrm{R}}(t) = \max_{F \in \Delta(\{\text{tensors}\})} F(t)$

Remark
$\underset{\sim}{\mathrm{Q}}(t)$ and $\underset{\sim}{\mathrm{R}}(t)$ are not in $\Delta(\{\text{tensors}\})$!

Remark
Theorem still holds when $\Delta(\{\text{tensors}\})$ is replaced by $\Delta(S)$
where $S \subseteq \{\text{tensors}\}$ closed under $\otimes, \oplus$ and $\langle 1 \rangle \in S$

Known: gauge points (Strassen 1986)

Transform tensor into matrix and take **matrix rank**

$$\zeta_1 : \{\text{tensors}\} \to \mathbb{R}_{\geq 0}$$

$$\left(t_{i_1 i_2 i_3}\right)_{i_1 i_2 i_3} \mapsto \operatorname{rank}\left(t_{i_1 (i_2, i_3)}\right)_{i_1 (i_2, i_3)}$$

Theorem (observation)

$$\zeta_1,\ \zeta_2,\ \zeta_3 \in \Delta(\{\text{tensors}\})$$

$$= \{F : \{\text{tensors}\} \to \mathbb{R}_{\geq 0} : \text{monot}, \text{add}, \text{mult}, \text{norm}\}$$

Known: support functionals for "oblique tensors" (Strassen 1986)

Study **probability distributions on the support** of $t \in \mathbb{C}^{n_1 \times n_2 \times n_3}$

$$\operatorname{supp} t = \left\{(i_1, i_2, i_3) : t_{i_1 i_2 i_3} \neq 0\right\} \subseteq [n_1]\times[n_2]\times[n_3]$$

oblique tensor means $\operatorname{supp} t$ is an antichain

$$\theta_1, \theta_2, \theta_3 \in \mathbb{R}_{\geq 0} \quad \theta_1 + \theta_2 + \theta_3 = 1$$

$$\begin{aligned} \zeta_\theta : \{\text{oblique tensors}\} &\to \mathbb{R}_{\geq 0} \\ t &\mapsto \max_{P \in \mathcal{P}(\operatorname{supp} t)} 2^{\theta_1 H(P_1) + \theta_2 H(P_2) + \theta_3 H(P_3)} \end{aligned}$$

Theorem

$$\zeta_\theta \in \Delta(\{\text{oblique tensors}\})$$

$$= \{F : \{\text{oblique tensors}\} \to \mathbb{R}_{\geq 0} : \text{monot, add, mult, norm}\}$$

New: quantum functionals

New: quantum functionals

Study **marginal density matrices**

• $s \in \mathbb{C}^{n_1 \times n_2 \times n_3}$, $\|s\|_2 = 1$	pure tripartite quantum state
• $\rho = ss^\dagger \in \mathbb{C}^{(n_1 \times n_2 \times n_3) \times (n_1 \times n_2 \times n_3)}$	density matrix
• $\rho_1 = \mathrm{Tr}_{2,3}\, \rho \in \mathbb{C}^{n_1 \times n_1}$	marginals
• $H(\rho_1)$	quantum entropy

$\theta_1, \theta_2, \theta_3 \in \mathbb{R}_{\geq 0} \quad \theta_1 + \theta_2 + \theta_3 = 1$

$F_\theta : \{\text{tensors}\} \to \mathbb{R}_{\geq 0}$

$$t \mapsto \sup\{2^{\theta_1 H(\rho_1) + \theta_2 H(\rho_2) + \theta_3 H(\rho_3)} : \rho = ss^\dagger;\ s \leq t;\ \|s\|_2 = 1\}$$

Main theorem (Christandl–Vrana–Zuiddam 2017)

$F_\theta \in \Delta(\{\text{tensors}\}) = \{F : \{\text{tensors}\} \to \mathbb{R}_{\geq 0} : \text{monot}, \text{add}, \text{mult}, \text{norm}\}$

Progress: Explicit points in the asymptotic spectrum of tensors

[S '86]	Gauge points	$\zeta_1, \zeta_2, \zeta_3 \in \Delta(\{\text{tensors}\})$
[S '86]	Support functionals	$\zeta_\theta \in \Delta(\{\text{oblique tensors}\})$
[CVZ '17]	Quantum functionals	$F_\theta \in \Delta(\{\text{tensors}\})$

$\theta_1, \theta_2, \theta_3 \in \mathbb{R}_{\geq 0}$
$\theta_1 + \theta_2 + \theta_3 = 1$

Relations

1. $\zeta_1 = F_{(1,0,0)}$ $\quad$ $\zeta_2 = F_{(0,1,0)}$ $\quad$ $\zeta_3 = F_{(0,0,1)}$
2. $\zeta_\theta = F_\theta$ on oblique tensors

See our paper for more relations

Crucial connection: Entanglement polytopes

$$F_\theta(t) = \sup\{2^{\theta_1 H(\rho_1)+\theta_2 H(\rho_2)+\theta_3 H(\rho_3)} : \rho = ss^\dagger; s \leq t; \|s\|_2 = 1\}$$

$$\Big\downarrow \; r_i = \mathrm{spec}(\rho_i)$$

$$F_\theta(t) = \sup\{2^{\theta_1 H(r_1)+\theta_2 H(r_2)+\theta_3 H(r_3)} : r \in \mathrm{E}_t\}$$

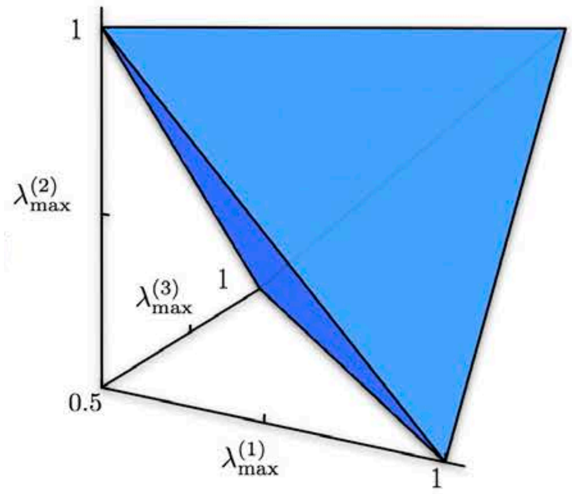
$$\mathrm{E}_t = \{\big(\mathrm{spec}(\rho_1), \mathrm{spec}(\rho_2), \mathrm{spec}(\rho_3)\big) : \rho = ss^\dagger; s \leq t; \|s\|_2 = 1\}$$

Crucial connection: Entanglement polytopes

$F_\theta(t) = \sup\{2^{\theta_1 H(r_1) + \theta_2 H(r_2) + \theta_3 H(r_3)} : r \in \mathrm{E}_t\}$

$\mathrm{E}_t = \{(\mathrm{spec}(\rho_1), \mathrm{spec}(\rho_2), \mathrm{spec}(\rho_3)) : \rho = ss^\dagger; s \leq t; \|s\|_2 = 1\}$

E_t is the **entanglement polytope** of t



[Walter–Doran–Gross–Christandl'13, Sawicki–Oszmaniec–Kuś'14] based on [Ness–Mumford'84, Brion'87]

Crucial connection: Entanglement polytopes

$$F_\theta(t) = \sup\{2^{\theta_1 H(r_1)+\theta_2 H(r_2)+\theta_3 H(r_3)} : r \in \mathrm{E}_t\}$$

Representation theory

$\lambda \vdash N$ is an integer partition of N

$P_\lambda^{\mathbb{C}^n} : (\mathbb{C}^n)^{\otimes N} \to$ "GL_n-isotypic component of type λ"

$$\mathrm{E}_t = \text{Euclidean closure of}$$
$$\left\{\left(\frac{\lambda}{N}, \frac{\mu}{N}, \frac{\nu}{N}\right) : \lambda, \mu, \nu \vdash N;\ \left(P_\lambda^{\mathbb{C}^{n_1}} \otimes P_\mu^{\mathbb{C}^{n_2}} \otimes P_\nu^{\mathbb{C}^{n_3}}\right) \cdot t^{\otimes N} \neq 0\right\}$$

Connection leads to key ingredients for our proof:
entropy inequalities of **Kronecker** and **Littlewood-Richardson coefficients**

Maximal values of the quantum functional

$F_\theta \leq n_1^{\theta_1} n_2^{\theta_2} n_3^{\theta_3}$ on $\mathbb{C}^{n_1 \times n_2 \times n_3}$

Theorem [Christandl–Vrana–Zuiddam 2017]
$\theta_1, \theta_2, \theta_3 > 0$
The maximal value of F_θ equals $n_1^{\theta_1} n_2^{\theta_2} n_3^{\theta_3}$

iff

$\mathbb{C}^{n_1 \times n_2 \times n_3}$ contains a pure quantum state with completely mixed marginals

Bryan, Reichstein, and Van Raamsdonk characterized such formats (n_1, n_2, n_3) in Monday QIP talk!

Conclusion

- $\gtrsim$ $\underset{\sim}{\mathrm{R}}(t)$ $\underset{\sim}{\mathrm{Q}}(t)$
  - algebraic complexity theory
  - property testing
  - combinatorics
  - communication complexity
  - asymptotic slocc

- characterized by the **asymptotic spectrum of tensors** $\Delta(\{\text{tensors}\})$

- We construct a nontrivial family of points in $\Delta(\{\text{tensors}\})$ using quantum information methods: **quantum functionals** F_θ

- **Are these all elements of** $\Delta(\{\text{tensors}\})$?
  (If yes, then matrix multiplication exponent ω equals 2.)