Quantum computing and Holant problems

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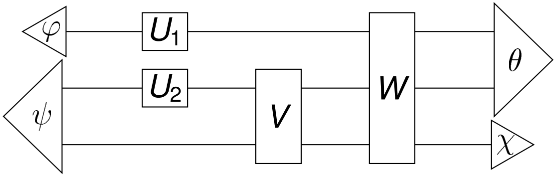
Strong classical simulation of quantum circuits



$$a = \langle\theta| \langle\chi| W(I \otimes V)(U_1 \otimes U_2 \otimes I) |\varphi\rangle |\psi\rangle$$

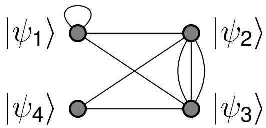
STRONG QUANTUM CIRCUIT SIMULATION(S)

- Input: a closed quantum circuit over S
- Output: the corresponding amplitude $a \in \mathbb{C}$

where:

- S is a set of operations: state preparations, unitary gates, and measurement projections
- all coefficients are algebraic complex numbers

Holant problems



$$\text{Holant} = \left(\bigotimes_{e \in E} \langle 00| + \langle 11| \right) \left(\bigotimes_{v \in V} |\psi_v\rangle \right)$$

HOLANT($\mathcal{S}$)

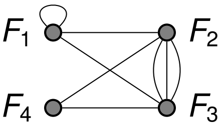
- Input: a graph, and an assignment of states from $\mathcal{S}$ to vertices
- Output: the value of the Holant

where

- $\mathcal{S}$ is a set of quantum states with algebraic complex coefficients
- each edge is projected onto $\langle 00| + \langle 11|$

This is a generalisation of STRONG QUANTUM CIRCUIT SIMULATION.

Holant problems



Holant

HOLANT($\mathcal{S}$)

- Input: a tensor network using tensors from $\mathcal{S}$
- Output: the scalar value of the tensor network contraction, Holant

where

- $\mathcal{S}$ is a set of tensors taking algebraic complex values
- all systems are qubits

Holant_Ω is the contraction of the tensor network.

Basic counting complexity theory

Definition

#P is the class of function problems of the form ‘compute $f(x)$’, where f is the number of accepting paths of a nondeterministic Turing machine running in polynomial time.

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Examples of #P-hard problems:

- STRONG QUANTUM CIRCUIT SIMULATION($\mathcal{S}$) for a universal set of operations $\mathcal{S}$
- counting (perfect) matchings of graphs
- counting vertex covers of graphs

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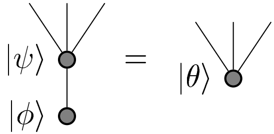
A dichotomy theorem states that all problems from a certain family of counting problems are either $\#$P-hard or can be solved in polynomial time.

Reduction techniques for Holant problems

The complexity of HOLANT $(\mathcal{S})$ is unaffected by the following operations:

- Adding gadgets to the given set of states $\mathcal{S}$:



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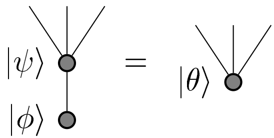

- Certain holographic transformations of the set $\mathcal{S}$: symmetric SLOCC operations, performed identically on all qubits in all states in $\mathcal{S}$, e.g.

$$
\begin{aligned}
\text{Holant} &= \left(\bigotimes_{e\in E} \langle 00| + \langle 11|\right)\left(\bigotimes_{v\in V} O^{\otimes|E(v)|}\,|\psi_v\rangle\right)\\
&= \left(\bigotimes_{e\in E} (\langle 00| + \langle 11|)(O\otimes O)\right)\left(\bigotimes_{v\in V} |\psi_v\rangle\right)\\
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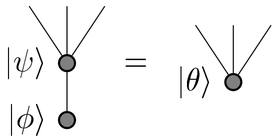

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- Adding states arising by polynomial interpolation to the set $\mathcal{S}$.

Complexity results about Holant problems

- HOLANT $(\mathcal{S})$
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  - polynomial-time computable cases are characterised by the entanglement properties of the states in $\mathcal{S}$

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Approach for deriving the $\textsc{Holant}^+$ dichotomy

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There exists a dichotomy for $\textsc{Holant}\left(\{\,|\psi\rangle\,\} \,\middle|\, \{\,|\phi\rangle\,\}\right)$, where $|\psi\rangle$ is a symmetric three-qubit state and $|\phi\rangle$ is a symmetric two-qubit state [Cai, Huang, Lu 2012].

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Given a set $\mathcal{S}$:

- If $\text{HOLANT}(\mathcal{S} \cup \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\})$ is known to be solvable in polynomial time, need to do nothing. This includes the case where $\mathcal{S}$ contains no multipartite entangled states.

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- Otherwise, there is multipartite entanglement.
  - Use gadgets to realise a symmetric entangled three-qubit state $|\psi\rangle$ and a symmetric entangled two-qubit state $|\phi\rangle$.
  - Then use the bipartite dichotomy to show hardness.

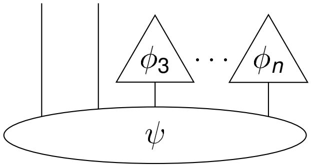
Assumptions:

- All the polynomial-time computable cases are known.
- If the problem is hard, we can show this via the bipartite dichotomy.

Realising small entangled states from large ones

Theorem (Popescu & Rohrlich 1992; Gachechiladze & Gühne 2017)

Let $|\psi\rangle$ be an *n*-system entangled state. For any two of the systems, there exists a projection onto a tensor product of states of the other $(n-2)$ systems that leaves the two systems in an entangled state.



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Theorem

Let $|\psi\rangle$ be an *n*-qubit entangled state with $n \geq 3$. Then there exists

- some choice of three qubits, and
- a projection of the other $(n-3)$ qubits onto a tensor product of $|0\rangle, |1\rangle, |+\rangle$ and $|-\rangle$

that leaves the chosen three qubits in an entangled state.

Proof sketch for three-qubit entanglement

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- This can be shown to lead to a contradiction. $\square$

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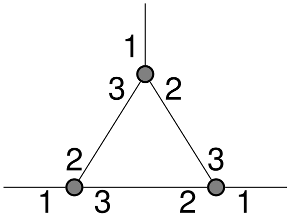
Gadget for a symmetric entangled three-qubit state



With a bit of work based on the entanglement classification of three-qubit states, can show:

Lemma

Given a set $\mathcal{S}$ containing an entangled three-qubit state

- either $\textsc{Holant}^+(\mathcal{S})$ can be solved in polynomial time, or
- it is possible to realise a symmetric entangled three-qubit state.

Can also produce symmetric entangled two-qubit states.

The complexity classification for HOLANT$^+$

Theorem

Let $\mathcal{S}$ be a set of quantum states with algebraic complex coefficients. Then

$$\text{HOLANT}^+(\mathcal{S}) := \text{HOLANT}\big(\mathcal{S} \cup \{\, |0\rangle\,, |1\rangle\,, |+\rangle\,, |-\rangle\,\}\big)$$

is polynomial time computable if

- the closure of $\mathcal{S} \cup \{\, |0\rangle\,, |1\rangle\,, |+\rangle\,, |-\rangle\,\}$ under taking gadgets contains:
  - only tensor products of one- and two-qubit states, or
  - GHZ-type entanglement but no *W*-type entanglement, or
  - *W*-type entanglement but no GHZ-type entanglement; or
- $\mathcal{S}$ contains only stabiliser states (up to scalar factors).

In all other cases, HOLANT$^+(\mathcal{S})$ is $\#$P-hard.

The real-valued HOLANT^c dichotomy

Theorem (Cai, Lu, Xia 2017)

Let $\mathcal{S}$ be a set of quantum states with algebraic real-valued coefficients. Then HOLANT$^c(\mathcal{S})$ is $\#$P-hard unless $\mathcal{S}$ is a tractable family for HOLANT* or for $\#\mathrm{CSP}_2^c$, where

$$\#\mathrm{CSP}_2^c(\mathcal{S}) = \mathrm{HOLANT}\left(\mathcal{S} \cup \{ \,|0\rangle\,, |1\rangle\, \} \cup \{ \,|\mathrm{GHZ}_{2n}\rangle \mid n \in \mathbb{N}\}\right).$$

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Proof (sketch).

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- Realise entangled states of reduced arity via gadgets with $|0\rangle, |1\rangle$, and self-loops ($|00\rangle + |11\rangle$). Then:

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- Either, can realise some ternary entangled state of a specific form. Hardness follows by various lemmas, some of which work only for real values.
- Or can realise or interpolate $\lvert \mathrm{GHZ}_4\rangle$, in which case the problem is equivalent to $\#\mathrm{CSP}_2^c(\mathcal{S})$, for which a full dichotomy (for complex coefficients) is derived in the same paper. □

Approach for complex-valued $\textsc{Holant}^c$

Combine methods from $\textsc{Holant}^+$ dichotomy proof with methods from real-valued $\textsc{Holant}^c$ dichotomy.

- Pick a multipartite entangled state and reduce arity using $|0\rangle$, $|1\rangle$ and $|00\rangle + |11\rangle$, as in the real-valued $\textsc{Holant}^c$ dichotomy.

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- If this results in an arbitrary fully entangled ternary state, proceed as in the HOLANT$^+$ dichotomy:

In some cases, additional single-qubit states may be required in the process; these can always be realised by gadgets.

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- If this results in an arbitrary fully entangled ternary state, proceed as in the HOLANT^+ dichotomy:
  - either show problem is easy,
  - or can reduce from a hard case of $\text{HOLANT}\left(\{|\psi\rangle\} \mid \{|\varphi\rangle\}\right)$.

  In some cases, additional single-qubit states may be required in the process; these can always be realised by gadgets.

Approach for complex-valued HOLANT^c

Combine methods from HOLANT$^+$ dichotomy proof with methods from real-valued HOLANT^c dichotomy.

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  In some cases, additional single-qubit states may be required in the process; these can always be realised by gadgets.
- Otherwise, prove we can realise or interpolate $|GHZ_4\rangle$. Then use the equivalence to $\#CSP_2^c(\mathcal{S})$ to show hardness, as in the real-valued HOLANT^c dichotomy.

The complexity classification for $\textsc{Holant}^c$

Theorem

Let $\mathcal{S}$ be a set of quantum states with algebraic complex coefficients. Then

$$\textsc{Holant}^c(\mathcal{S}) := \textsc{Holant}(\mathcal{S} \cup \{|0\rangle, |1\rangle\})$$

is polynomial time computable if

- the closure of $\mathcal{S} \cup \{|0\rangle, |1\rangle\}$ under taking gadgets contains:
  - only tensor products of one- and two-qubit states, or
  - GHZ-type entanglement but no W-type entanglement, or
  - W-type entanglement but no GHZ-type entanglement; or
- $\mathcal{S}$ contains only stabiliser states (up to scalar factors and certain SLOCC operations).
- $\mathcal{S}$ contains only states $|\psi\rangle$ with the following property: let n be the number of qubits in $|\psi\rangle$, then for all bit strings $x_1 \ldots x_n$ such that $\langle x_1 \ldots x_n|\psi\rangle \neq 0$,

  $$(T^{x_1} \otimes \ldots \otimes T^{x_n})\,|\psi\rangle$$

  is (up to scalar factor) a stabiliser state, where $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$.

In all other cases, $\textsc{Holant}^c(\mathcal{S})$ is $\#$P-hard.

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Open questions:

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Thank you!

The polynomial-time computable families, part 1

- $\mathcal{U}$ denotes the set of all single-qubit states
- $|\mathrm{GHZ}_n\rangle = |0\rangle^{\otimes n} + |1\rangle^{\otimes n}$
- $|W_n\rangle = |1\rangle |0\rangle^{\otimes n-1} + |0\rangle |1\rangle |0\rangle^{\otimes n-2} + \ldots + |0\rangle^{\otimes n-1} |1\rangle$
- $\langle \mathcal{S} \rangle$ denotes the closure of the set $\mathcal{S}$ under taking gadgets

HOLANT $(\mathcal{S})$ is polynomial-time computable if $\mathcal{S}$...

1. contains only tensor products of one- and two-qubit states
2. contains only stabiliser states
3. contains GHZ-type entanglement but no W-type entanglement, i.e.

$$\mathcal{S} \subseteq \left\langle \left\{ |\mathrm{GHZ}_n\rangle : n \in \mathbb{N} \right\} \cup \{|01\rangle + |10\rangle\} \cup \mathcal{U} \right\rangle$$

4. contains W-type entanglement but no GHZ-type entanglement: after applying $\left(\begin{smallmatrix} 1 & \pm i \\ 1 & \mp i \end{smallmatrix}\right)$ to each qubit in each state in $\mathcal{S}$, get a subset of

$$\left\langle \left\{ |W_n\rangle : n \in \mathbb{N} \right\} \cup \left\{ |00\rangle + c\,|11\rangle : c \in \mathbb{C} \right\} \cup \mathcal{U} \right\rangle$$

5. satisfies property 2 or 3 after certain symmetric SLOCC operations

The polynomial-time computable families, part 2

HOLANT$^+$ $(\mathcal{S})$ can be solved in polynomial time if all $|\psi\rangle \in \mathcal{S}$ have the following property: let n be the number of qubits in $|\psi\rangle$, then for all bit strings $x_1 \ldots x_n$ such that $\langle x_1 \ldots x_n|\psi\rangle \neq 0$,

$$(T^{x_1} \otimes \ldots \otimes T^{x_n}) |\psi\rangle$$

is (up to scalar factor) a stabiliser state, where $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$.

Examples:

$\|\psi\rangle$	$\|00\rangle + \|11\rangle$	$\|01\rangle + \|10\rangle$	$\|01\rangle + e^{i\pi/4}\|10\rangle$
xy s.t. $\langle xy\|\psi\rangle \neq 0$	00, 11	01, 10	01, 10
$(T^0 \otimes T^0)\|\psi\rangle$	$\|00\rangle + \|11\rangle$	–	–
$(T^0 \otimes T^1)\|\psi\rangle$	–	$e^{i\pi/4}\|01\rangle + \|10\rangle$	$e^{i\pi/4}(\|01\rangle + \|10\rangle)$
$(T^1 \otimes T^0)\|\psi\rangle$	–	$\|01\rangle + e^{i\pi/4}\|10\rangle$	$\|01\rangle + i\|10\rangle$
$(T^1 \otimes T^1)\|\psi\rangle$	$\|00\rangle + i\|11\rangle$	–	–
property satisfied?	yes	no	yes