Rigorous free fermion entanglement renormalization from wavelet theory

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joint work with J. Haegeman, G. Evenbly, J. Cotler, B. Swingle, M. Walter

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Results

We construct tensor networks for **free fermion systems**

- For fermions hopping on 1 & 2 D lattices
- Rigorous approximation guarantees

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Key features:

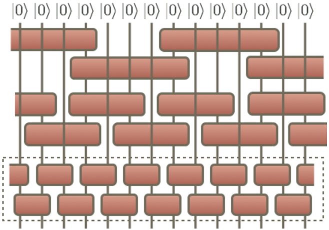
- tensor networks that target **correlation functions**
- **quantum circuits** that ‘renormalize entanglement’: MERA
- **explicit** circuit construction, no variational optimization required

Outline

- Entanglement renormalization (MERA)
- Wavelet transforms
- Rigorous entanglement renormalization for fermions
- Outlook & Summary

Entanglement renormalization (MERA)

MERA: multi-scale entanglement renormalization ansatz (Vidal)



↓ local **quantum circuit** that prepares state from $|0\rangle^{\otimes N}$

↑ entanglement renormalization disentangle local degrees of freedom

↕ organize q. information by scale

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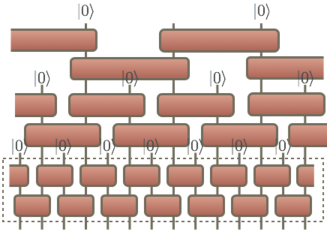

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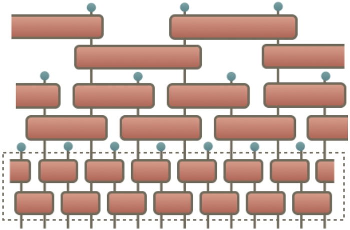

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↕ **organize q. information by scale**

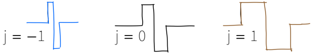
- layers are short-depth quantum circuits (disentangle & coarse-grain)
- variational class for **critical systems** in 1D
- any MERA can be extended to a 'holographic' mapping (reminiscent of holography (Swingle))

Wavelet transforms

Wavelets

Wavelet transforms resolve **classical signal** into different scales

- multi-resolution analysis: $L^2(\mathbb{R}) = \bigoplus_j W_j$, spanned by $\psi(2^{-j}x - n)$
- ψ is called the *wavelet function*



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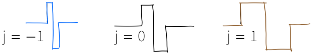
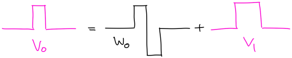


Given signal at scale **up to** j, $V_j = \bigoplus_{j' \geq j} W_{j'}$, how to resolve it into scales?

- V_j spanned by $\phi(2^{-j}x - n)$,
- ϕ is known as *scaling function*



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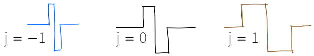

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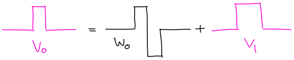
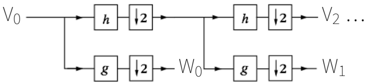


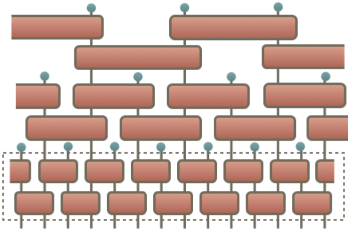
Discrete wavelet transform:



- defined by low-pass filter h and high-pass filter g
- locally resolves discrete input signal in $\ell^2(\mathbb{Z})$ into different scales

MERA and wavelets

Key fact: Second quantizing 1D wavelet transform $\rightsquigarrow$ MERA circuit!



length of classical filter ~ depth of quantum circuit (Evenbly-White)

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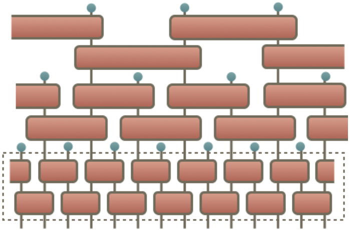

length of classical filter ~ depth of quantum circuit (Evenbly-White)

Task: To produce free fermion ground state, design wavelet transform that targets positive/negative energy modes.

Rigorius entanglement renormalization

1D Dirac fermions – Lattice model

Massless Dirac fermions on 1D lattice (Kogut-Susskind):

$$H_{1D} = -\sum_n b^\dagger_{1,n} b_{2,n} - b^\dagger_{2,n} b_{1,n+1} + b^\dagger_{2,n} b_{1,n} - b^\dagger_{1,n+1} b_{2,n}$$

$$= \int_{-\pi}^{\pi} \frac{dk}{2\pi} \begin{bmatrix} b_1(k) \\ b_2(k) \end{bmatrix}^\dagger \begin{bmatrix} 0 & e^{-ik} - 1 \\ e^{ik} - 1 & 0 \end{bmatrix} \begin{bmatrix} b_1(k) \\ b_2(k) \end{bmatrix}.$$

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Diagonalize:

$$u(k) = \begin{bmatrix} 1 & 0 \\ 0 & -i\,\mathrm{sign}(k) e^{ik/2} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad u^\dagger h u = \begin{bmatrix} E_-(k) & 0 \\ 0 & E_+(k) \end{bmatrix}$$

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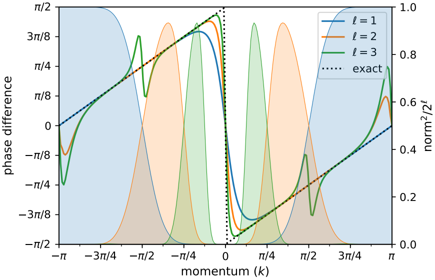
- freedom to choose *any* basis of Fermi sea!
- want **pairs** of modes related by $-i\,\mathrm{sign}(k) e^{ik/2}$.

1D Dirac fermions – Wavelets

Task: Find pair of wavelet transforms such that **high-pass filters** are related by $-\mathrm{i}\,\mathrm{sign}(k)e^{\mathrm{i}k/2}$.

- studied in signal processing, motivated by *translation-invariance*
- impossible with finite filters, but possible to arbitrary accuracy

  (Selesnick)



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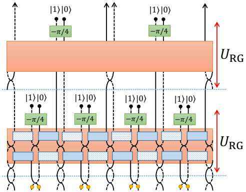

Parameters:

- $\mathcal{L}$ – number of layers
- ε – accuracy of phase relation of filters
- W – "size" of filters

Consider **correlation function** of N creation and annihilation operators

$$C(\{f_i\}) := \left\langle b^\dagger_{j_1}(f_1)\cdots b^\dagger_{j_N}(f_N)\, b_{j_{N+1}}(f_{N+1})\cdots b_{j_{2N}}(f_{2N})\right\rangle$$

supported on S lattice sites.

Theorem (simplified)

$$\left|C(\{f_i\})_{\mathrm{exact}} - C(\{f_i\})_{\mathrm{MERA}}\right| \lesssim \sqrt{S}NW\max\{2^{-\mathcal{L}/4}, \varepsilon\}$$

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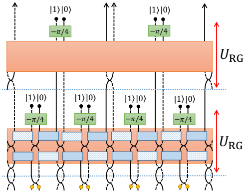

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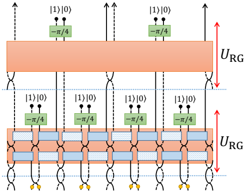

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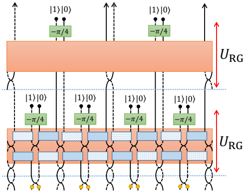

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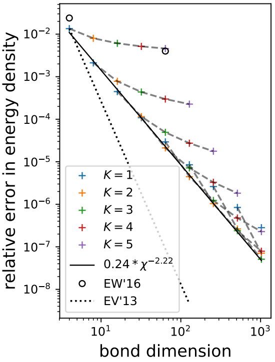
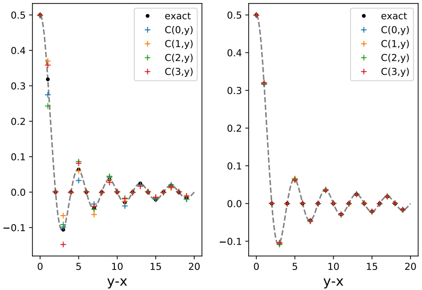
1D Dirac fermions – Numerics

Energy error



Green function $C(x, y) = \langle a_x^\dagger a_y \rangle$



bond dimension 2^2

bond dimension 2^6

Non-relativistic 2D fermions – Lattice model

Non-relativistic fermions hopping on 2D square lattice at half filling:

$$H_{2D} = -\sum_{m,n} a^\dagger_{m,n} a_{m+1,n} + a^\dagger_{m,n} a_{m,n+1} + \text{h.c.}$$

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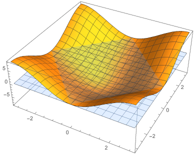
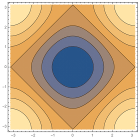
Fermi surface:





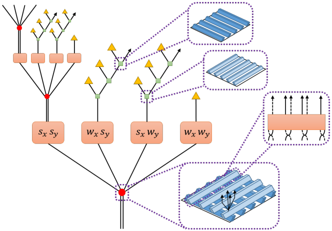
- violation of area law: $S(R) \sim R \log R$ (Wolf, Gioev-Klich, Swingle)
- Green function factorizes w.r.t. rotated axes

Non-relativistic 2D fermions – Branching MERA

Natural construction: Tensor product of wavelet transforms!

$$W\psi = \psi_S \oplus \psi_W \quad \rightsquigarrow \quad (W \otimes W)\psi = \psi_{SS} \oplus \psi_{WS} \oplus \psi_{SW} \oplus \psi_{WW}$$

After second quantization, obtain variant of branching MERA (Evenbly-Vidal):



Similar approximation theorem holds.

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- Massive theories, Dirac cones, beyond states at fixed times, …
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