Moderate deviation analysis for c-q channels (and hypothesis testing)

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QIP 2018
QuTech, TU Delft

Two techniques

Refined small-deviation analysis:

- *"Moderate deviation analysis for classical communication over quantum channels"*, **Christopher T. Chubb**, Vincent Y.F. Tan, and Marco Tomamichel, Communications in Mathematical Physics (2017) 355: 1283, arXiv:1701.03114.

Refined large-deviation analysis:

- *"Moderate Deviation Analysis for Classical-Quantum Channels and Quantum Hypothesis Testing"*, **Hao-Chung Cheng** and Min-Hsiu Hsieh, IEEE Transactions on Information Theory (to appear), arXiv:1701.03195.
- *"Quantum Sphere-Packing Bounds with Polynomial Prefactors"*, **Hao-Chung Cheng**, Min-Hsiu Hsieh, and Marco Tomamichel, arXiv:1704.05703.

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Suppose Alice wants to send classical information to Bob, via some channel.

There is a trade-off between three quantities:

- Number of channel uses
- Amount of information transmitted
- Error probability

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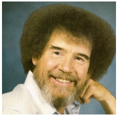

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  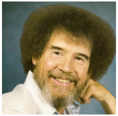

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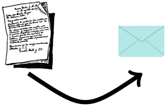

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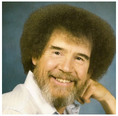

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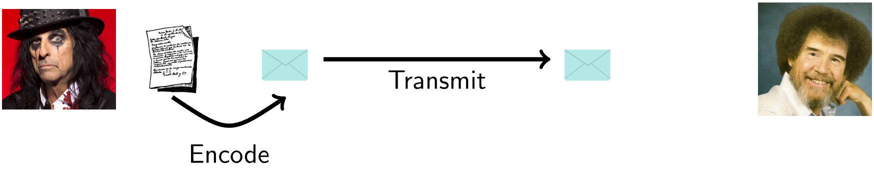

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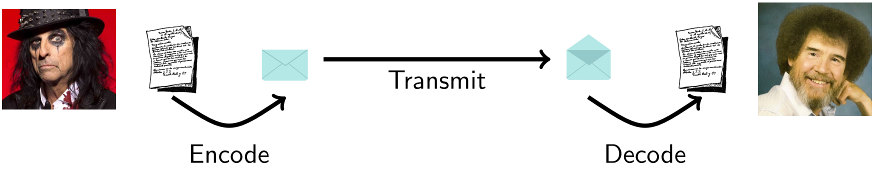

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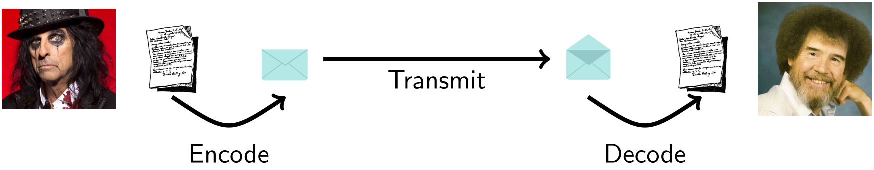

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Classical channels versus quantum channels

If we have access to a quantum channel, then quantum encoding/decoding can allow us to transmit more information with less error than classical encoding/decoding.

A simple example is a bit-flip channel

$$\mathcal{E}(\rho) = pX\rho X + (1-p)\rho.$$

Classically: Either we send many noisy bits, or fewer encoded bits.

Quantumly: Simply transmit our bits noiselessly in the X basis $\{|+\rangle, |-\rangle\}$.

Classical communication over a quantum channel

We are going to consider coding of classical-quantum channels.

For c-q channel $\mathcal{W}$, a (n, R, ϵ)-code is an encoder E and decoding POVM $\{D_i\}$ such that

$$\frac{1}{2^{nR}} \sum_{m=1}^{2^{nR}} \mathrm{Tr}\left[\mathcal{W}^{\otimes n}\left(\otimes_{i=1}^{n} E_i(m)\right) D_m\right] \geq 1 - \epsilon$$

We will be concerned with the trade-off between the block-length n, the rate R, and the error probability ϵ. We define the optimal rate/error probability as

$$\begin{aligned} R^*(\mathcal{W}; n, \epsilon) &:= \max\left\{R \mid \exists (n, R, \epsilon)\text{-code}\right\}, \\ \epsilon^*(\mathcal{W}; n, R) &:= \min\left\{\epsilon \mid \exists (n, R, \epsilon)\text{-code}\right\}. \end{aligned}$$

Asymptotics

For a constant error probability ϵ, the Strong Converse Theorem tells us the rate approaches a constant known as the capacity

$$\lim_{n\to\infty} R^*(\mathcal{W}; n, \epsilon) = C(\mathcal{W}).$$

Equivalently this means that the error probability must to go 0 to 1 either side of the capacity

$$\lim_{n\to\infty} \epsilon^*(\mathcal{W}; n, R) = \begin{cases} 0 & : R < C(\mathcal{W}) \\ 1 & : R > C(\mathcal{W}) \end{cases}$$

This tells us we can have either $R \to C$ OR $\epsilon \to 0$.

How fast are these convergences? Can we do both?

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Small and large deviations

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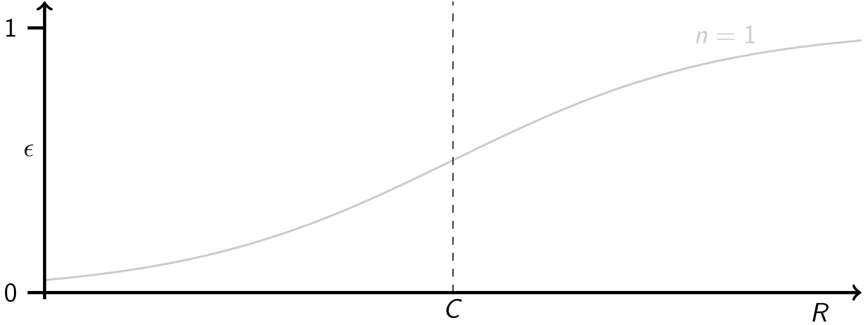

Small deviation (Tomamichel and Tan 2015)

$$R^*(n, \epsilon) = C + \sqrt{\frac{V}{n}}\Phi^{-1}(\epsilon) + o\left(\frac{1}{\sqrt{n}}\right) \quad \epsilon \in (0, \tfrac{1}{2})$$

Large deviation (Partial progress)

$$\ln \epsilon^*(n, R) = -n \cdot E(R) + o(n) \quad R < C$$

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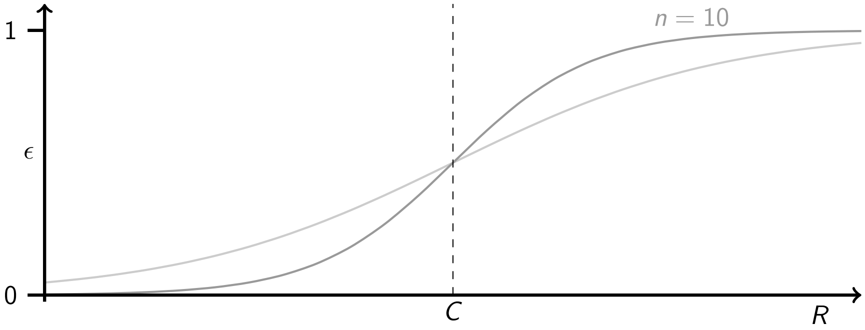

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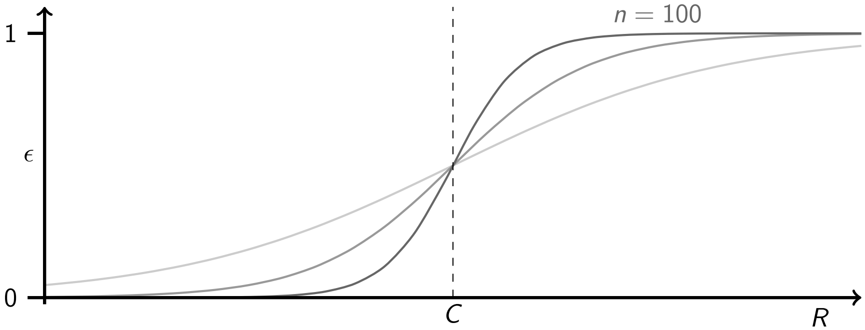

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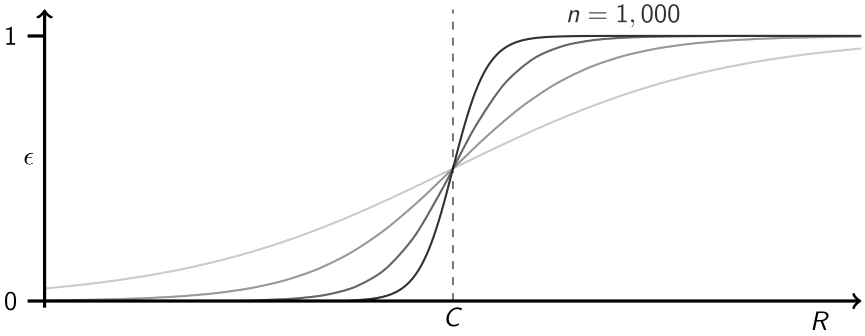

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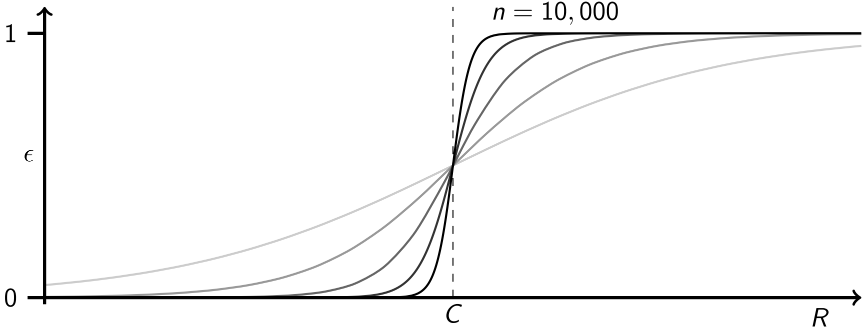

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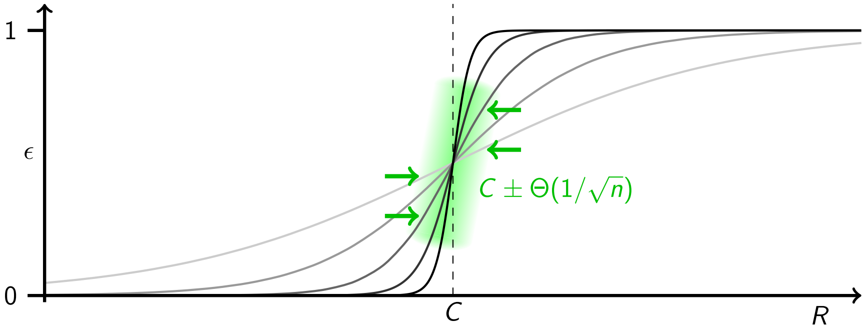

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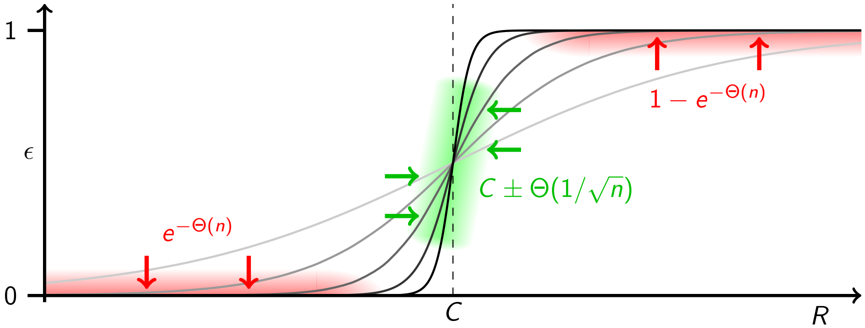

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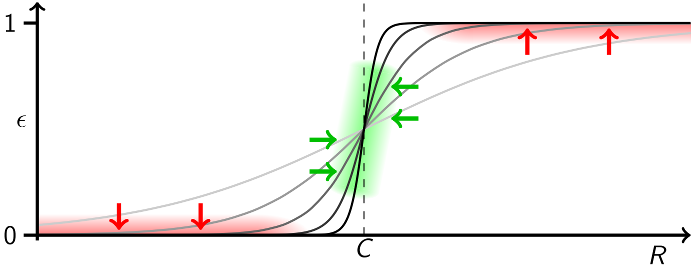

Moderate deviation (This work)

For any $\{a_n\}$ such that $a_n \to 0$ and $\sqrt{n}a_n \to \infty$ we have

$$R^*(n,\epsilon_n) = C - \sqrt{2V}a_n + o(a_n) \quad \text{for} \quad \epsilon_n = e^{-na_n^2},$$

or equivalently

$$\ln \epsilon^*(n,R_n) = -\frac{na_n^2}{2V} + o(na_n^2) \quad \text{for} \quad R_n = C - a_n.$$

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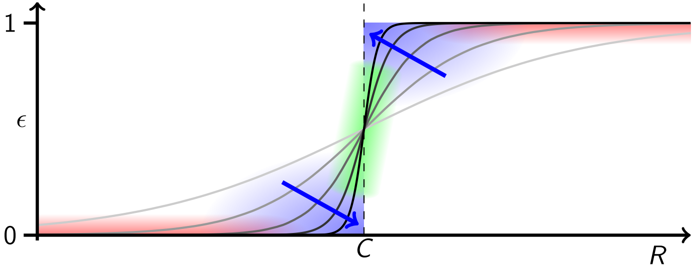

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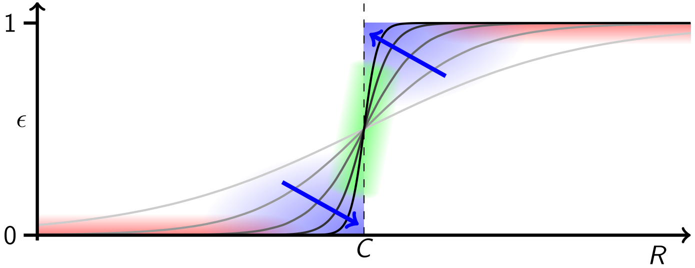

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Concentration inequalities

Take $\{X_i\}$ iid with $\mathbb{E}[X_i] = 0$ and $\mathrm{Var}[X_i] =: V$, and $\bar{X}_n := \frac{1}{n}\sum_{i=1}^n X_i$.

Asymptotic (Law of large numbers)

$$\lim_{n\to\infty} \Pr\left[\bar{X}_n \geq t\right] = \begin{cases} 1 & t < 0, \\ 0 & t > 0. \end{cases}$$

Small deviation (Berry-Esseen)

$$\Pr\left[\bar{X}_n \geq \frac{\epsilon}{\sqrt{n}}\right] = Q\left(\frac{\epsilon}{\sqrt{V}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \quad \epsilon \in (0,1)$$

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Hypothesis testing

We want to test between two hypotheses, ρ and σ. For a binary POVM $\{A, I-A\}$, we define the type-I and type-II errors as

$$\alpha(A;\rho,\sigma) := \operatorname{Tr}(I-A)\rho, \qquad \beta(A;\rho,\sigma) := \operatorname{Tr} A\sigma,$$

and the ϵ-hypothesis-testing divergence

$$D_h^\epsilon(\rho\|\sigma) := -\log \min_{0\le A\le I} \{\beta(A;\rho,\sigma) \,|\, \alpha(A;\rho,\sigma) \le \epsilon\}.$$

If we now consider testing between $\rho^{\otimes n}$ and $\sigma^{\otimes n}$, then the asymptotic behaviour is given by Quantum Stein's Lemma.

Asymptotics (Hiai and Petz 1991, Ogawa and Nagaoka 1999)

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Deviation results for hypothesis testing

Small deviation (Tomamichel and Hayashi 2013, Li 2014)

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Bounding the rate

For this we can use the one shot bounds

$$R^*(1,\epsilon) \geq \sup_{P_X} D_h^{\epsilon/2}(\pi_{XY}\|\pi_X\otimes\pi_Y) - \mathcal{O}(1), \qquad \text{(Wang and Renner 2012)}$$

$$R^*(1,\epsilon) \leq \inf_{\sigma} \sup_{\rho\in\mathrm{Im}(\mathcal{W})} D_h^{2\epsilon}(\rho\|\sigma) + \mathcal{O}(1), \qquad \text{(Tomamichel and Tan 2015)}$$

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Reducing hyp. testing to concentration inequalities

To give a moderate deviation analysis of the HTD, we will use concentration bounds. First we see it is related to tail bounds of the Nussbaum-Szkoła distributions[1]

$$P^{\rho,\sigma}(a,b) := r_a |\langle \phi_a | \psi_b \rangle|^2 \quad \text{and} \quad Q^{\rho,\sigma}(a,b) := s_b |\langle \phi_a | \psi_b \rangle|^2,$$

where we have eigendecomposed our states $\rho := \sum_a r_a |\phi_a\rangle\langle\phi_a|$ and $\sigma := \sum_b s_b |\psi_b\rangle\langle\psi_b|$. These reproduce the first two moments of our states

$$D\left(P^{\rho,\sigma} \| Q^{\rho,\sigma}\right) = D(\rho\|\sigma) \qquad \text{and} \qquad V\left(P^{\rho,\sigma} \| Q^{\rho,\sigma}\right) = V(\rho\|\sigma).$$

Specifically for iid $Z_i = \log P^{\rho,\sigma}/Q^{\rho,\sigma}$ and $(a_i, b_i) \sim P^{\rho,\sigma}$, then[2]

$$\frac{1}{n} D_h^{\epsilon_n}\left(\rho^{\otimes n} \middle\| \sigma^{\otimes n}\right) \geq \sup\left\{ R \,\middle|\, \Pr\left[\sum_{i=1}^n Z_i\right] \leq \epsilon_n/2 \right\} - \mathcal{O}(\log 1/\epsilon_n),$$

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[1] Nussbaum and Szkoła 2009.

[2] Tomamichel and Hayashi 2013

Reducing hyp. testing to concentration inequalities

To give a moderate deviation analysis of the HTD, we will use concentration bounds. First we see it is related to tail bounds of the Nussbaum-Szkoła distributions[1]

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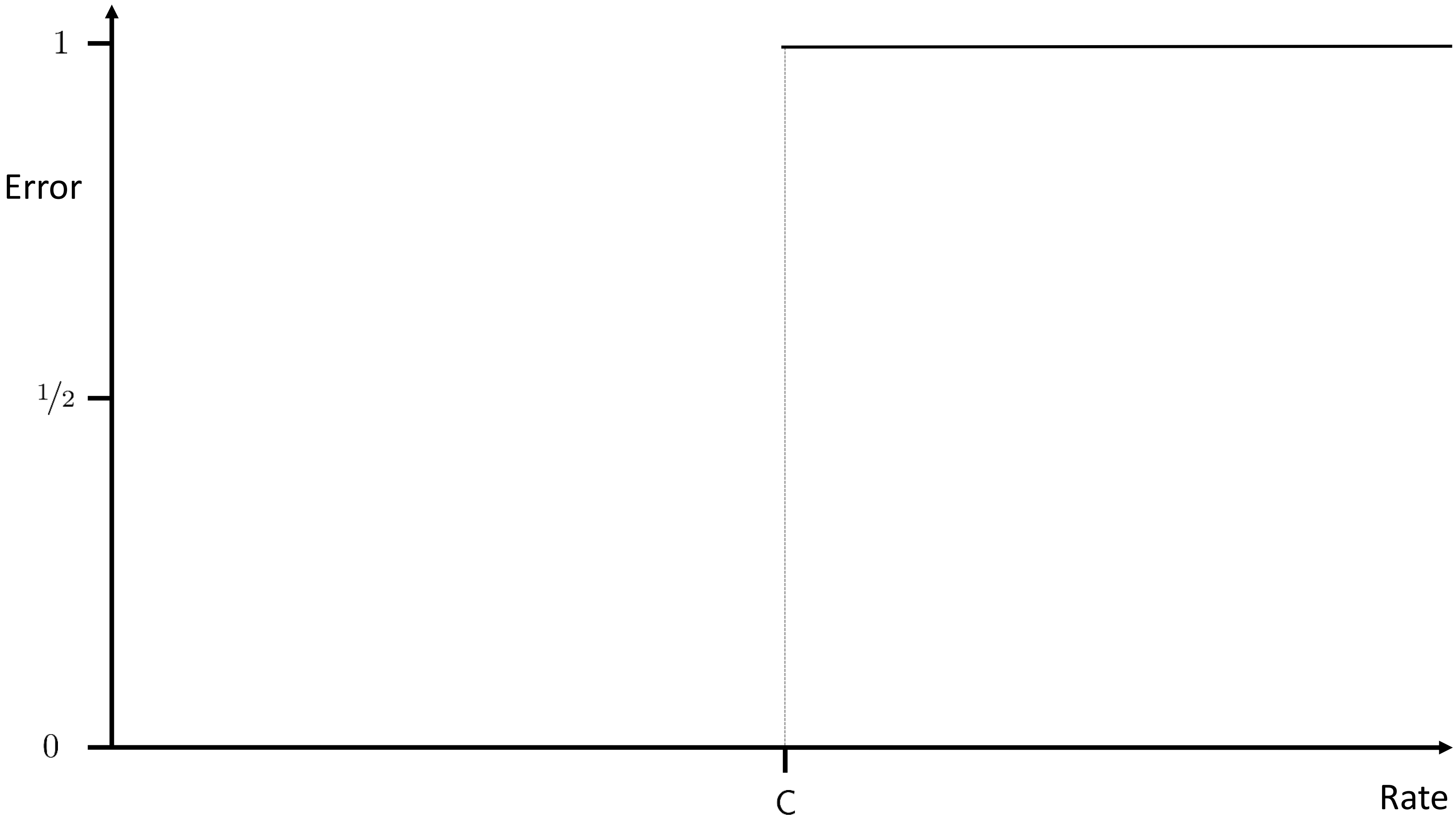
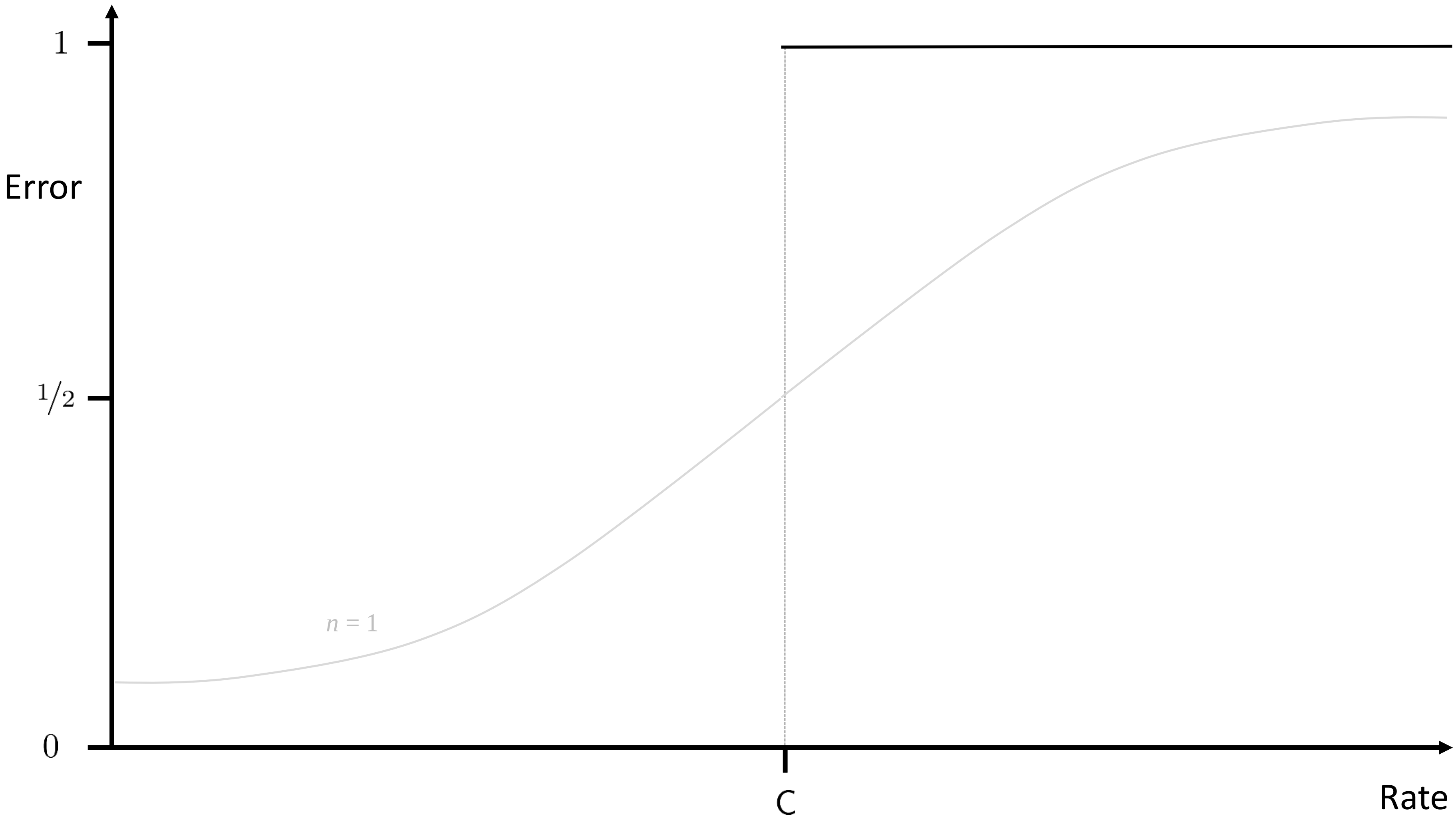
Different regimes



Different regimes



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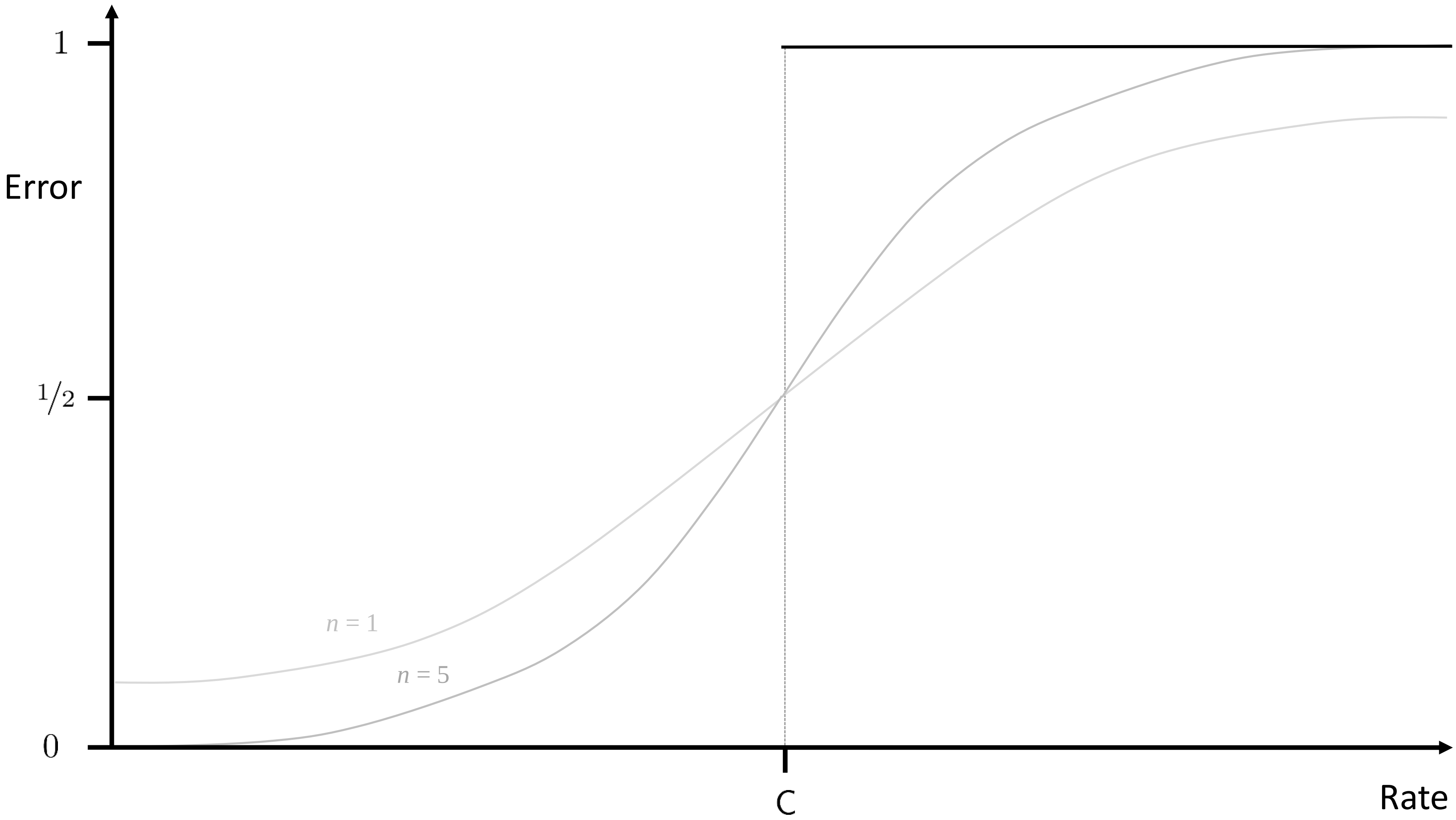

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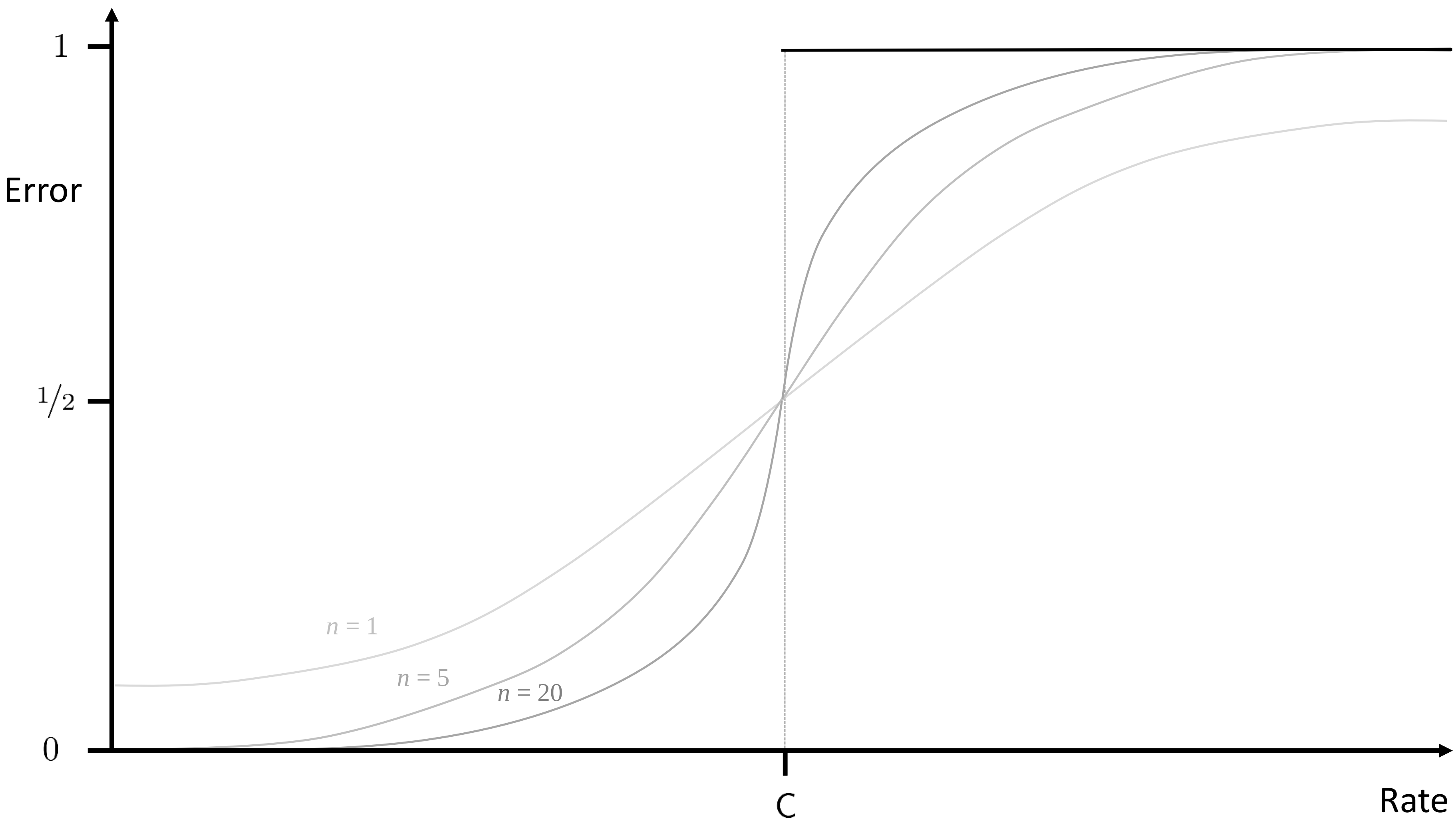

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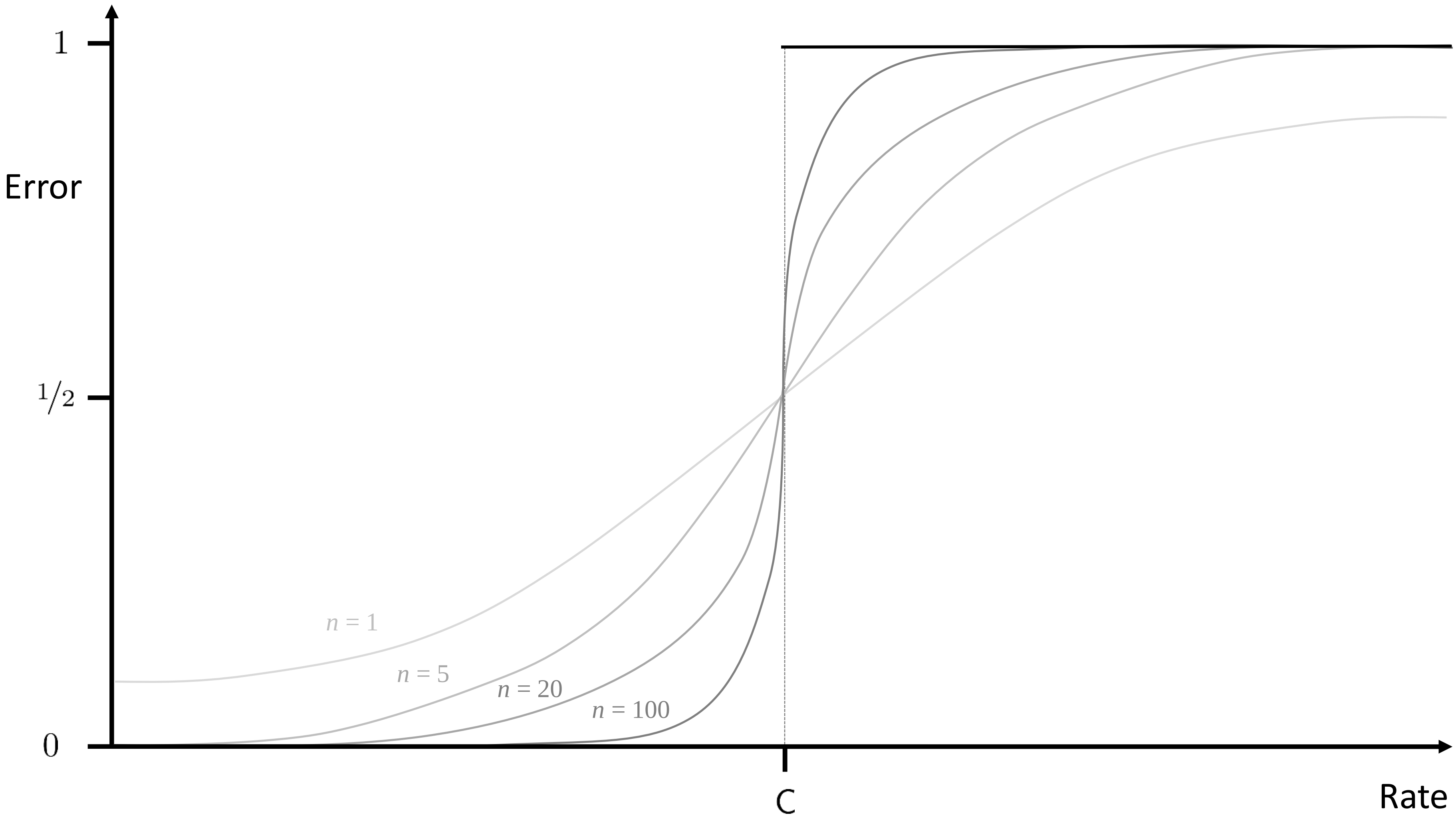

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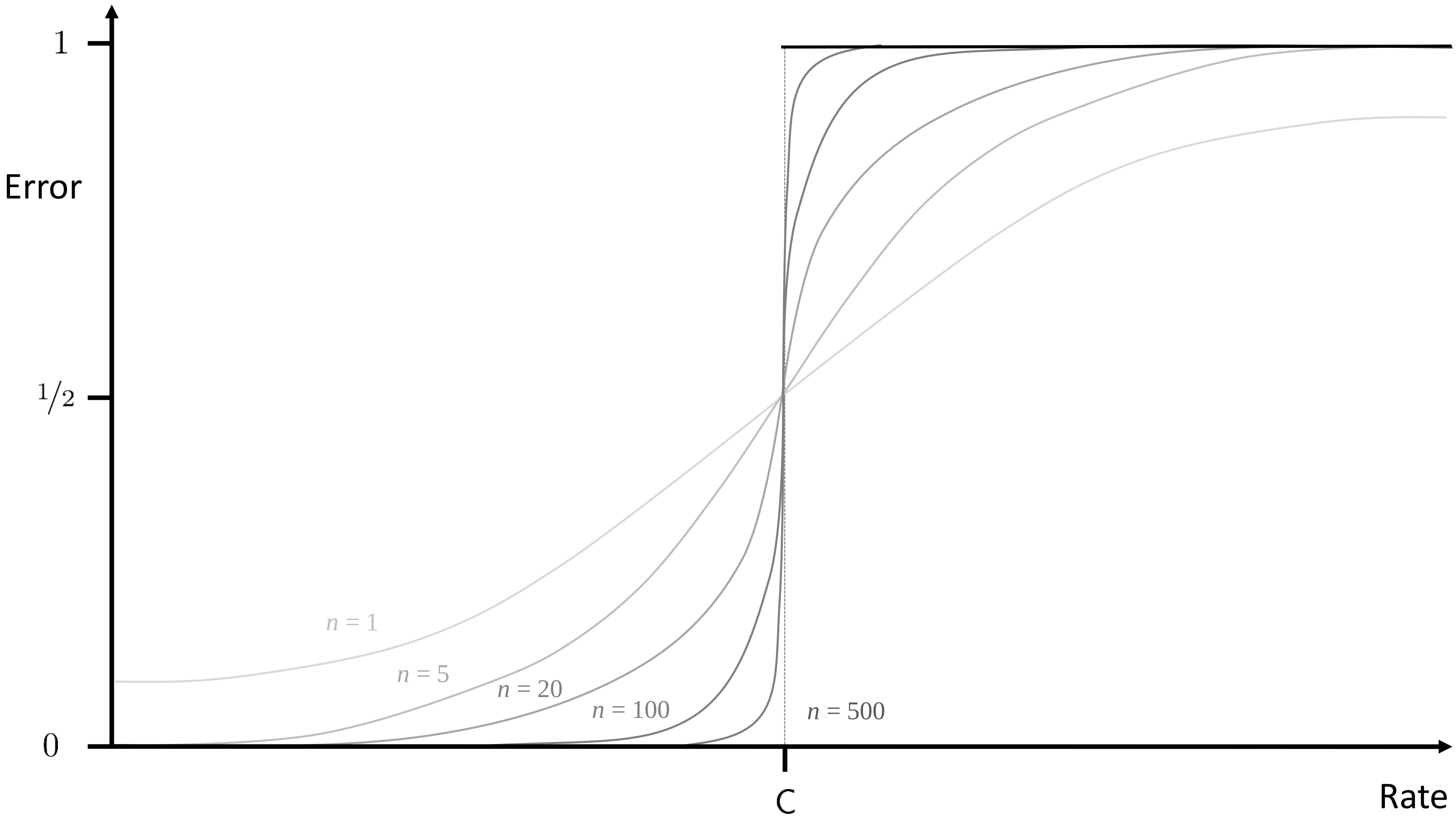

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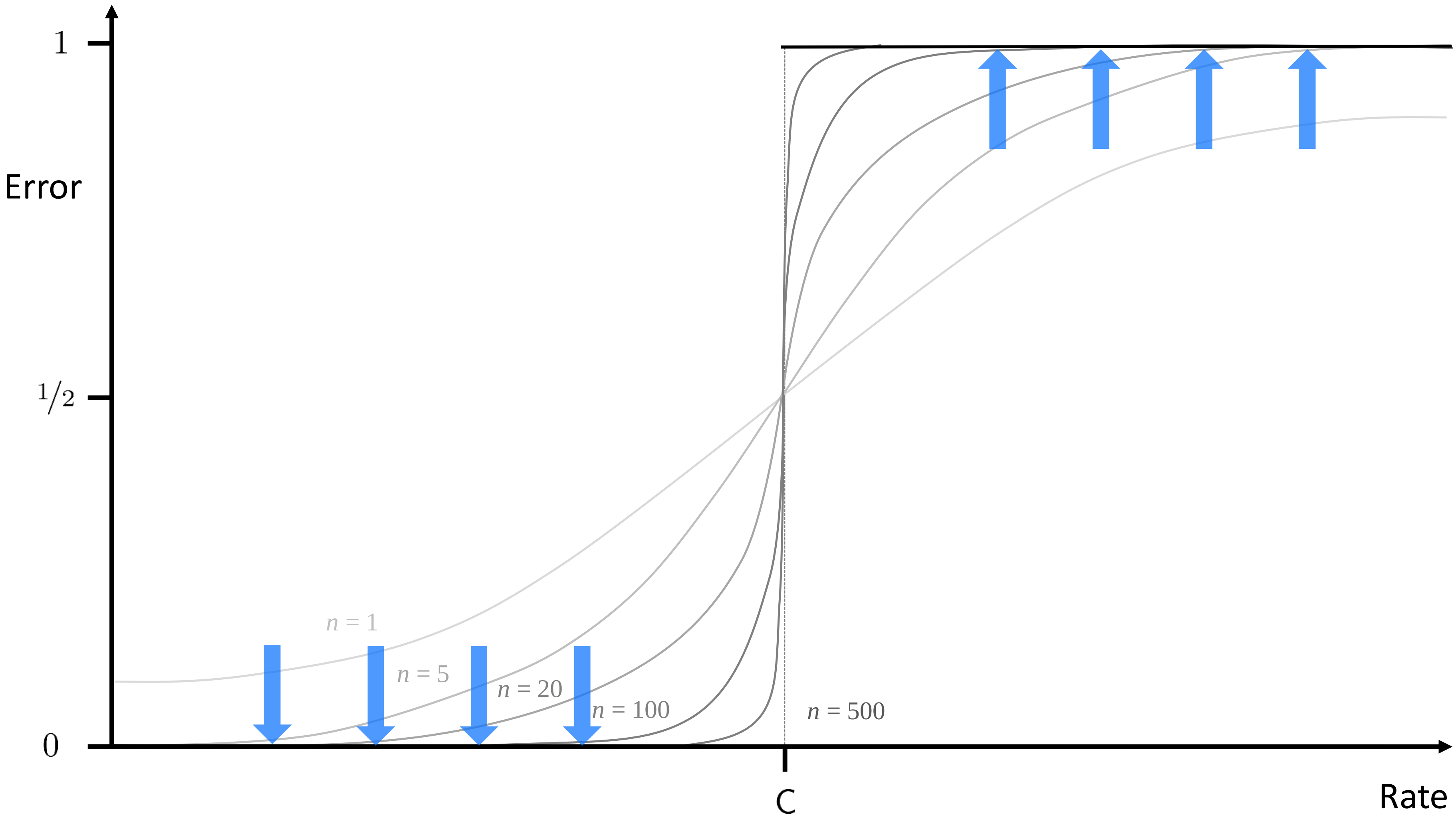

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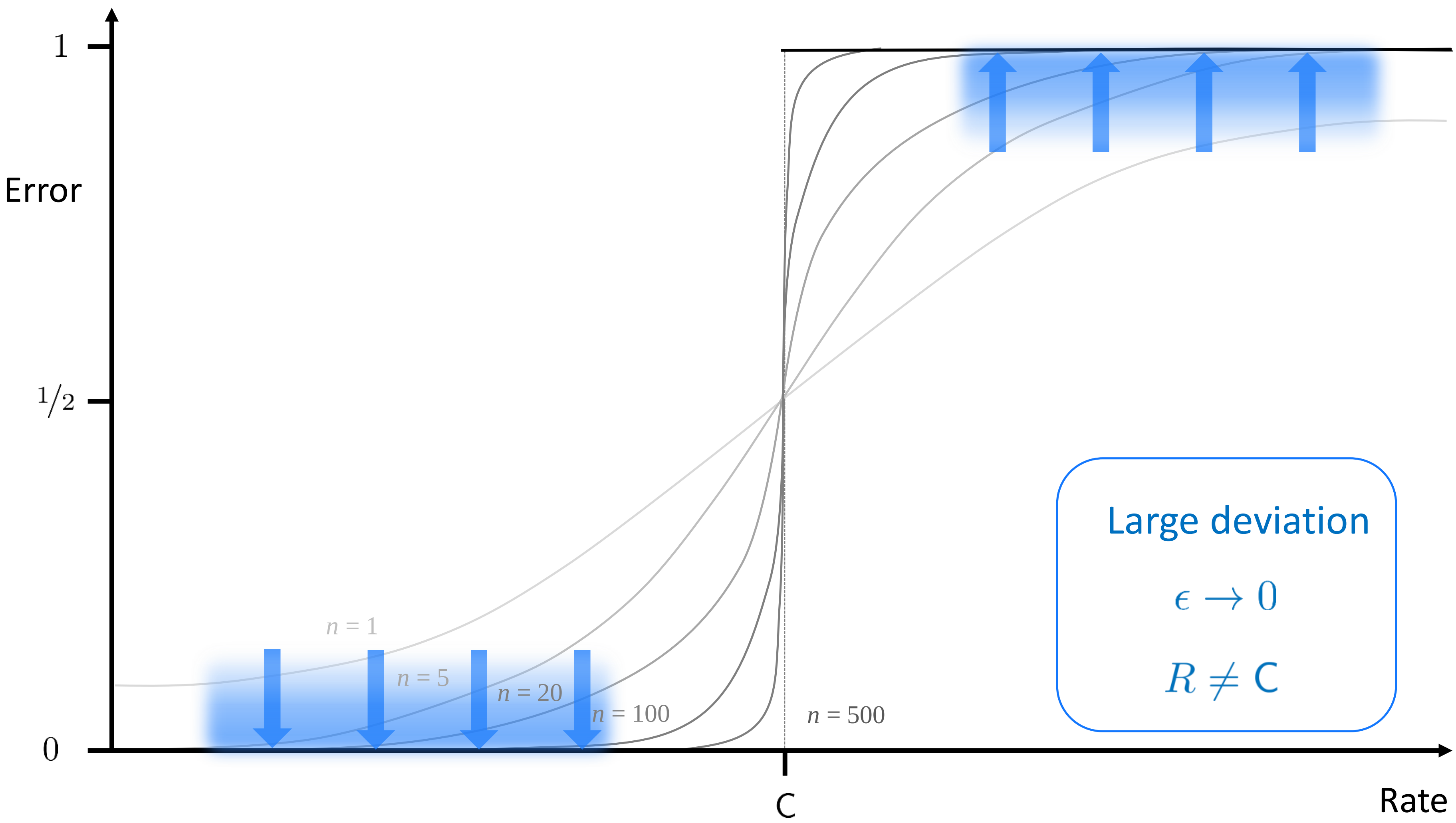

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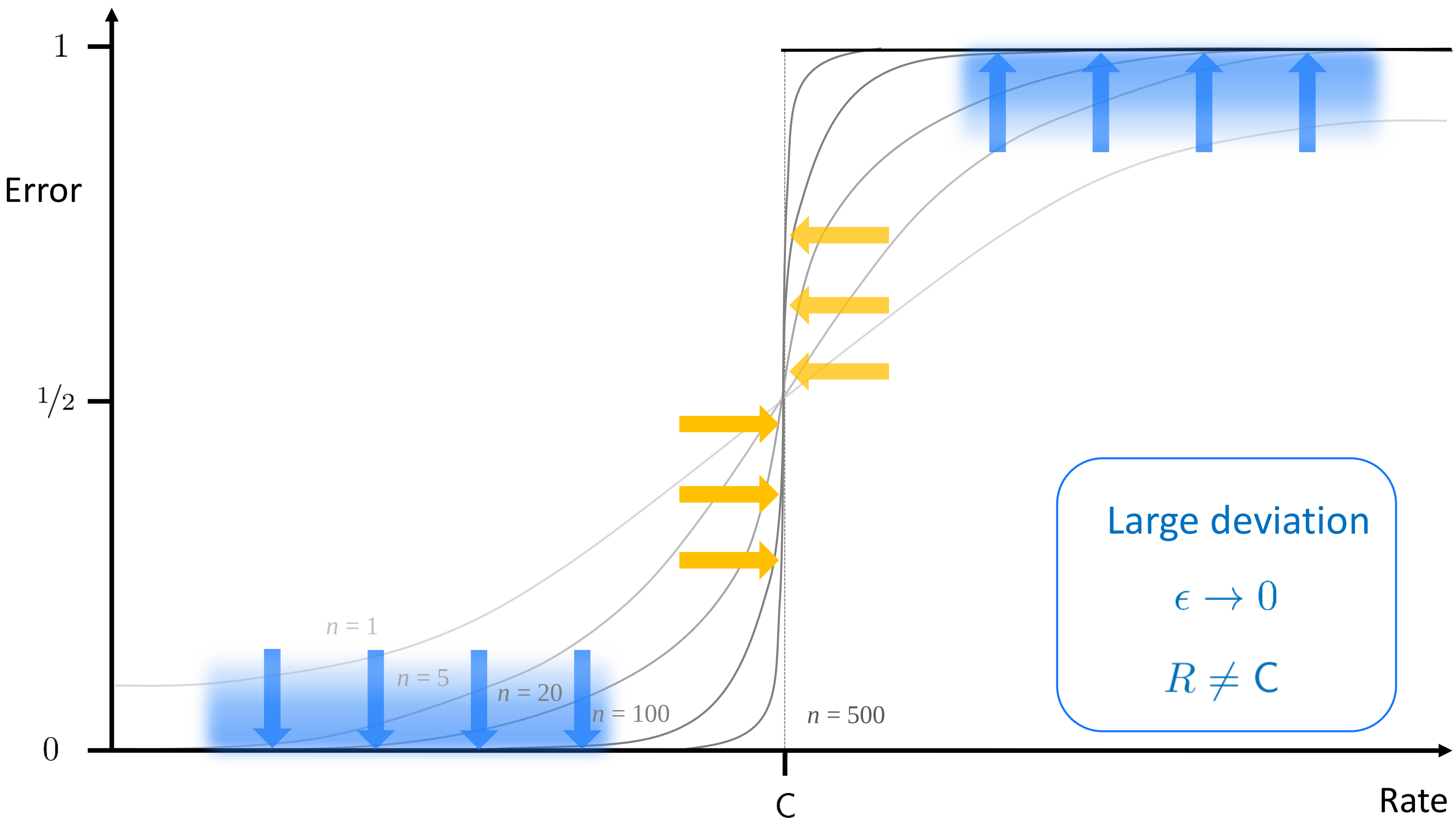

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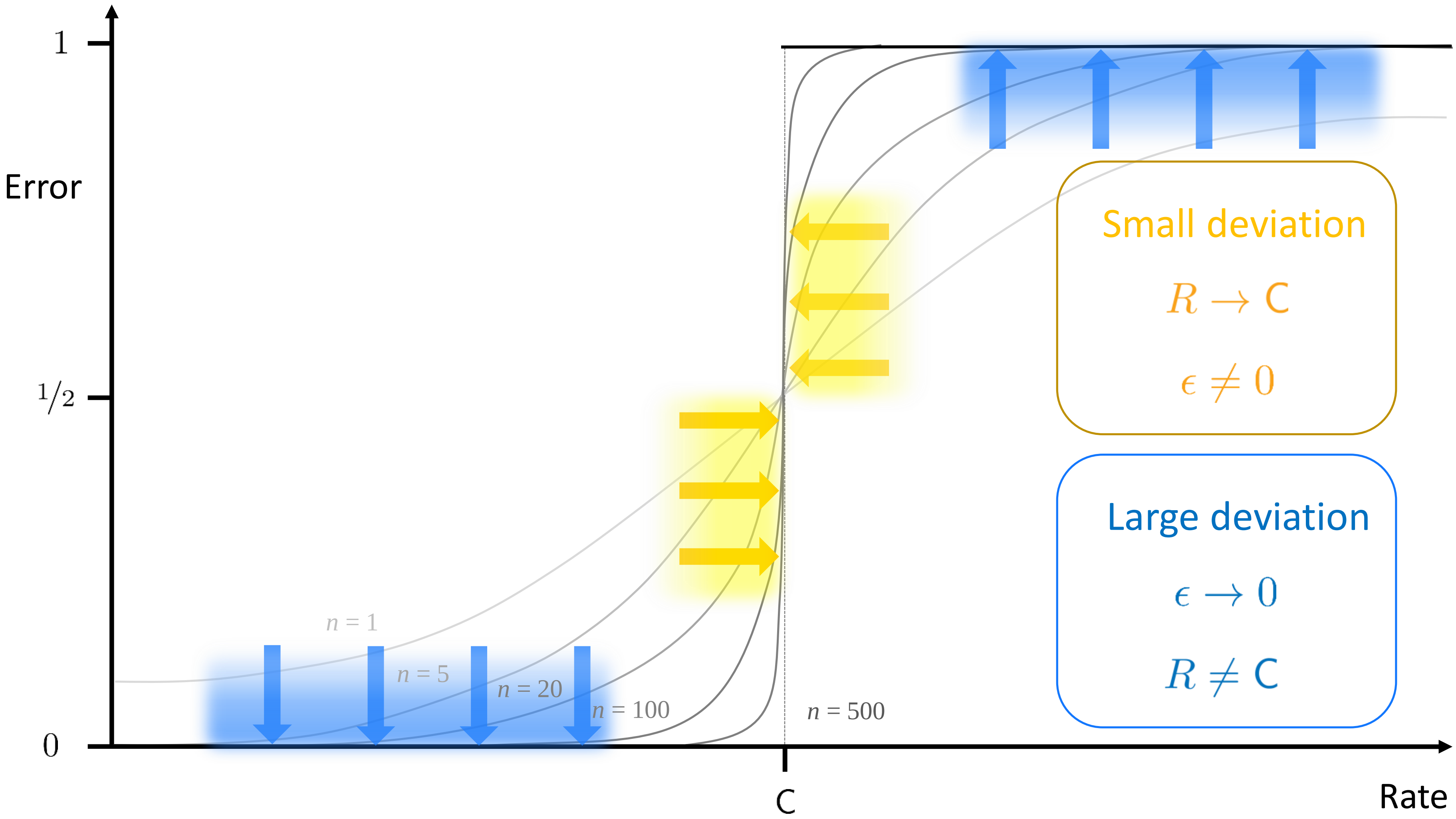

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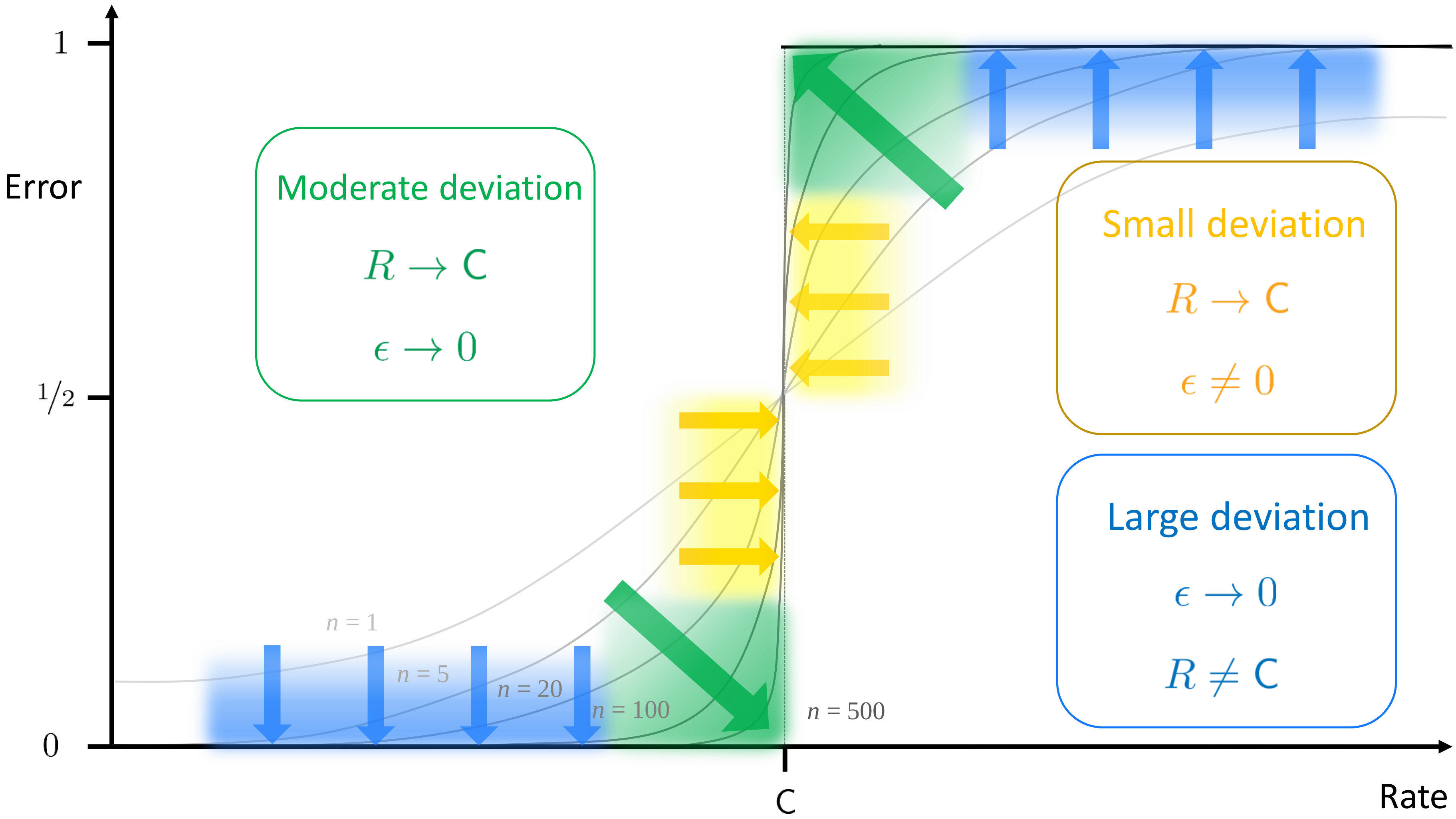

From large deviation regime

Q Is the reliable communication possible as rate approaches capacity?

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$$\Rightarrow \quad \boxed{\epsilon^*(n, R_n) = \exp\left\{ -\frac{n a_n^2}{2\mathsf{V}} + o(n a_n^2) \right\} \to 0}$$

Proof ideas

$$R^*(n, \epsilon_n) = \mathsf{C} - \sqrt{2\mathsf{V}} a_n + o(a_n)$$

$$\epsilon_n = \exp\{-na_n^2\}$$

Channel coding

$$\epsilon^*(n, R_n) = \exp\left\{-\frac{na_n^2}{2\mathsf{V}} + o(na_n^2)\right\}$$

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$\uparrow$

$$-\frac{1}{n}\log\beta_n^* \to D - \sqrt{2V}a_n,$$

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Hypothesis testing

$$\alpha_n^* \to \exp\left\{-\frac{na_n^2}{2V}\right\},$$

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[Small deviation]

Concentration inequalities

[Large deviation]

Moderate deviations for hypothesis testing

- Type-I, -II errors: $\alpha_n := \mathrm{Tr}\left[(\mathbb{1} - A_n)\rho^{\otimes n}\right]$

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- Given $\beta_n \leq \exp\{-nR\}$

- Quantum Stein's lemma (Hiai and Petz 1991, Ogawa and Nagaoka 1999)

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- Answer:

$$\boxed{\alpha_n^* = \exp\left\{ -\frac{n a_n^2}{2V(\rho\|\sigma)} + o(n a_n^2) \right\} \to 0}$$

Proof idea

- Quantum Hoeffding bound ($\beta_n \leq \exp\{-nR\}$)

$$\sup_{0<\alpha\leq 1} \frac{1-\alpha}{\alpha} \left(D_\alpha(\rho\|\sigma) - R \right)$$

$$\alpha_n^* = \exp\{-n\mathsf{E}(R) + o(n)\}$$

- Achievability (Audenaert *et al.* 2007, Hayashi 2007, Audenaert, Nussbaum, Szkola, Verstraete 2008)

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Channel coding

- Goal: for $R_n = \mathsf{C} - a_n$,

$$\Rightarrow \epsilon^*(n, R_n) = \exp\left\{ -\frac{n a_n^2}{2\mathsf{V}} + o(n a_n^2) \right\}$$

Information variance

$$\mathsf{V} := \sup_{\rho_X : I(X:B)_\rho = \mathsf{C}} V(\rho_{XB} \| \rho_X \otimes \rho_B)$$

- Challenges:
  - The optimal error exponent is still open
  - Need a tight finite blocklength analysis for the optimal error probability

Achievability

- Hayashi 2007: $\epsilon^*(n, R) \leq 4 \exp\left\{-n\mathsf{E}_{\mathrm{r}}^{\downarrow}(R)\right\}$

$$\max_{\frac{1}{2}\leq\alpha\leq 1} \frac{1-\alpha}{\alpha}\left(D_{2-\frac{1}{\alpha}}(\rho_{XB}\|\rho_X \otimes \rho_B) - R\right)$$

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Converse (Sphere-Packing Bound)

- Winter 1999:

$$\lim_{n\to\infty} -\frac{1}{n}\log \epsilon^*(n,R) \le \widetilde{\mathsf{E}}_{\mathrm{sp}}(R) := \max_{\rho_X} \min_{\sigma_{XB}:\sigma_X=\rho_X} \{D(\sigma_{XB}\|\rho_{XB}) : \mathsf{I}(X:B)_\sigma \le R\}$$

- Dalai 2013:

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- Questions:
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  - Finite blocklength bound with tight prefactor?

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- Classical approach (Altug, Wagner 2014)

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[arXiv:1704.05703]

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- We study the fundamental trade-off between error, rate, and blocklength
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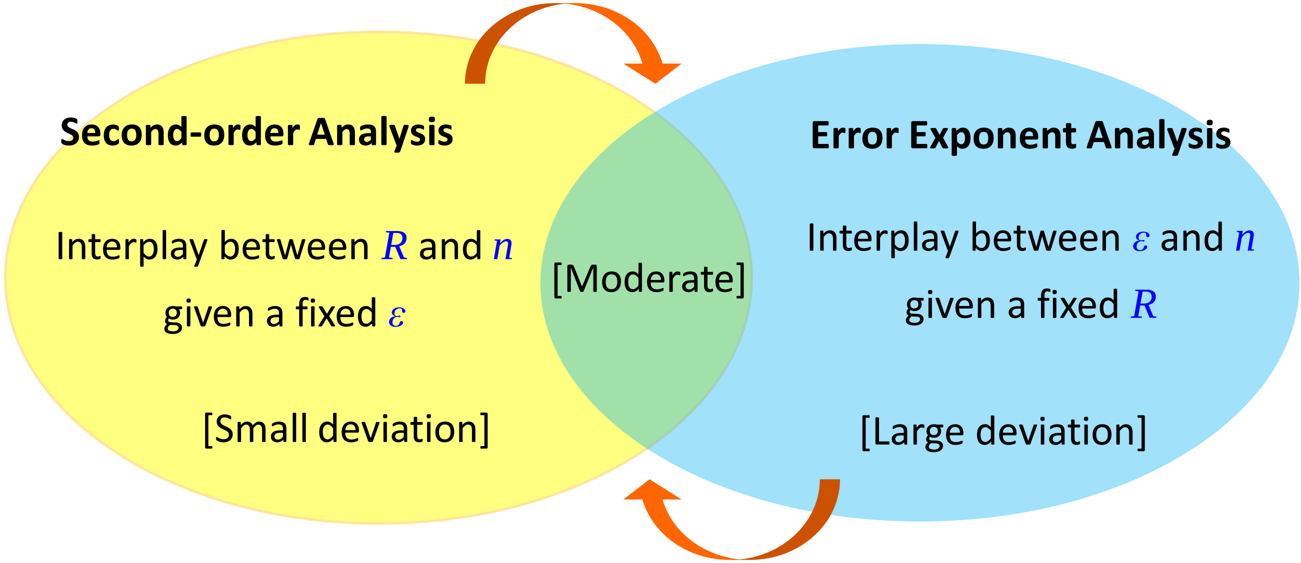
[Chubb, Tan, Tomamichel]



[Cheng, Hsieh]

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- Extension to image-additive channels – What about other channels (entanglement-breaking) or capacities (entanglement-assisted)?

- Other applications – private communications, classical data compression with quantum side information, *etc.*

Different concentration regimes

Regimes	Channel Coding	Concentration
Small deviation	$\epsilon^*\left(n, \mathsf{C} - \frac{A}{\sqrt{n}}\right) \sim \Phi\left(\frac{A}{\sqrt{\mathsf{V}}}\right)$	$\Pr\left[\bar{X}_n \geq \frac{1}{\sqrt{n}} t\right] \sim 1 - Q\left(\frac{x}{\sqrt{\mathsf{V}}}\right)$
Moderate deviation	$\epsilon^*(n, \mathsf{C} - a_n) = \mathrm{e}^{-\frac{n a_n^2}{2\mathsf{V}} + o(n a_n^2)}$	$\Pr\left[\bar{X}_n \geq a_n t\right] = \mathrm{e}^{-\frac{n a_n^2}{2\mathsf{V}} x + o(n a_n^2)}$
Large deviation	$\epsilon^*(n, R) = \mathrm{e}^{-n\mathsf{E}(R) + o(n)}$	$\Pr\left[\bar{X}_n \geq t\right] = \mathrm{e}^{-nI(x) + o(n)}$