The cost of destroying entanglement and resource

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Outline for section 1

The set up

ρ_{AB}

Groisman, Popescu, Winter [Phys. Rev. A., 2005]

The set up

ρ_{AB}

$\sigma_{A'}$

$\boxed{\mathrm{J}}$

$$\tau_J = \frac{1}{N}\sum_{j=1}^{N} |j\rangle\langle j|_{\mathrm{J}}$$

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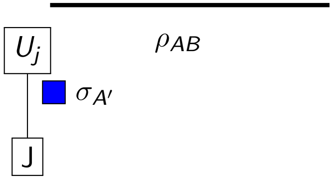
The set up



$$\tau_J = \frac{1}{N}\sum_{j=1}^{N} |j\rangle\langle j|_{\mathrm{J}}\,, \quad U_{AA'J} = \sum_j |j\rangle\langle j|_J \otimes U_j$$

Groisman, Popescu, Winter [Phys. Rev. A., 2005]

The set up

$\rho'_{AA'B}$

$$\rho'_{AA'B} = \mathrm{Tr}_J(U_{AA'J}\rho_{AB} \otimes \sigma_{A'} \otimes \tau_J U^\dagger_{AA'J})$$

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$$\rho'_{AA'B} \approx \rho_A \otimes \rho_B \otimes \sigma_{A'}$$

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- Showed that $\frac{1}{n} \log |J| \to \mathrm{I}(A : B)_\rho$.
- Achievability does not use A' register.
- $\mathrm{I}(A : B)_\rho = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$.
- $\mathrm{I}(A : B)_\rho = \mathrm{D}(\rho_{AB} \| \rho_A \otimes \rho_B) = \inf_{\sigma_A, \sigma_B} \mathrm{D}(\rho_{AB} \| \sigma_A \otimes \sigma_B)$.
- Measure of correlation.

Outline for section 2

Our set up

ρ_{AB}

The set up

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J — J'

$$\tau_{JJ'} = \frac{1}{N}\sum_{j=1}^{N} |j\rangle\langle j|_{\mathrm{J}} \otimes |j\rangle\langle j|_{\mathrm{J'}}$$

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$\sigma_{A'B'}$

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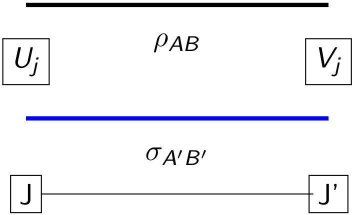

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$\omega_{AA'BB'}$

$$\omega_{AA'BB'} = \mathrm{Tr}_{JJ'}((U_{AA'J} \otimes V_{BB'J'})\rho_{AB} \otimes \sigma_{A'B'} \otimes \tau_{JJ'}(U^{\dagger}_{AA'J} \otimes V^{\dagger}_{BB'J'}))$$

The set up

$$\omega_{AA'BB'} \approx \theta_{AB} \otimes \sigma_{A'B'}$$

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θ_{AB} is separable

Some definitions

- Definitions:

$$E(\rho_{AB}) = \inf_{\sigma_{AB} \in SEP(A:B)} \mathrm{D}(\rho_{AB} \| \sigma_{AB})$$

$$E_{\varepsilon}^{one-shot}(\rho_{AB}) = \inf_{\sigma_{AB} \in SEP(A:B)} \mathrm{D}_{\max}^{\varepsilon}(\rho_{AB} \| \sigma_{AB})$$

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- $E(\rho_{AB})$: relative entropy of entanglement. (Vedral, Plenio, Rippin, Knight [Phys. Rev. Lett. 1997]).
  - An entanglement measure.
  - Monotonic under non-entangling operations.
  - Natural interpretation under quantum hypothesis testing (Brandao, Plenio [Comm. Math. Phys., 2010]).

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- $\mathrm{D}_{\max}(\rho \| \sigma) = \inf\{k : \rho \preceq 2^k \sigma\}$ (Datta [IEEE TIT, 2009]).
- $\mathrm{D}_{\max}^{\varepsilon}(\rho \| \sigma) = \inf_{\rho' \overset{\approx}{\varepsilon} \rho} \mathrm{D}_{\max}(\rho \| \sigma)$.

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- $\mathrm{D}_{\max}^{\varepsilon}(\rho\|\sigma) = \inf_{\rho' \overset{\approx}{\varepsilon} \rho} \mathrm{D}_{\max}(\rho\|\sigma)$.
- Robustness of entanglement (Vidal, Tarrach [Phys. Rev. A., 1999]).

Our results

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There exists a protocol with error $\varepsilon + \delta$ requires discarding $E_{\varepsilon}^{one-shot}(\rho_{AB}) + 2\log\frac{1}{\delta}$ bits. Furthermore any protocol with error ε must discard $E_{\varepsilon}^{one-shot}(\rho_{AB})$ bits.

- Error measured by the purified distance (Gilchrist, Langford, Nielsen [Phys. Rev. A., 2005]).

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- In the asymptotic and i.i.d. setting, this becomes $\lim_{n\to\infty}\frac{1}{n}E(\rho_{AB}^{\otimes n})$.
- Recall: $E(\rho_{AB}) = \inf_{\sigma_{AB}\in SEP(A:B)} \mathrm{D}(\rho_{AB}\|\sigma_{AB})$.
- Thus, disentanglement cost is an entanglement measure.

Outline for section 3

Locally recovered states

- Fix a tripartite quantum state ρ_{ABC}.
- The set of all locally recovered states are $(\mathrm{I}_B \otimes \mathcal{R}_{C\to AC})(\rho_{BC})$.
- Relative entropy of recovery:

$$\mathrm{D}(A:B|C)_\rho = \inf_{\mathcal{R}_{C\to AC}} \mathrm{D}(\rho_{ABC} \| (\mathrm{I}_B \otimes \mathcal{R}_{C\to AC})(\rho_{BC})).$$

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- Regularized version is $\mathrm{D}^\infty(A:B|C)_\rho$.
  - A natural interpretation as hypothesis testing between a quantum state and the set of locally recovered states. (Cooney, Hirche, Morgan, Olson, Seshadreesan, Watrous, Wilde [Phys. Rev. A., 2016]).
  - A lower bound on the quantum conditional mutual information. (Brandao, Harrow, Oppenheim, Winter [Phys. Rev. Lett., 2015]).

Transformation via local mixtures of unitaries

- Local mixture of unitaries is of the following form

$$\frac{1}{J}\sum_{i=1}^{J} U_A^i \otimes V_B^i \otimes W_C^i (.) U_A^{i\dagger} \otimes V_B^{i\dagger} \otimes W_C^{i\dagger}.$$

- How much noise ($\log J$) is required to convert ρ_{ABC} to a locally recovered state, allowing catalysts that are locally recovered states?

Characterization of the noise

- Achievability and a converse:

Theorem (MB/CM)

There exists a protocol that requires a rate of $\mathrm{D}^{\infty}(A:B|C)_{\rho}$ *bits in the asymptotic and i.i.d. setting. Furthermore any protocol for which the unitary* V_B^i *are permutations requires a rate of* $\mathrm{D}^{\infty}(A:B|C)_{\rho}$ *bits.*

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- The achievability protocol above has the property that V_B^i are permutation operations.
- Open question to find near optimal characterization under arbitrary local mixture of unitaries.

Outline for section 4

Resource theories basics

- Free states given by $\mathcal{F}$. We assume:
  - Convex set.
  - Closed under tensor product.
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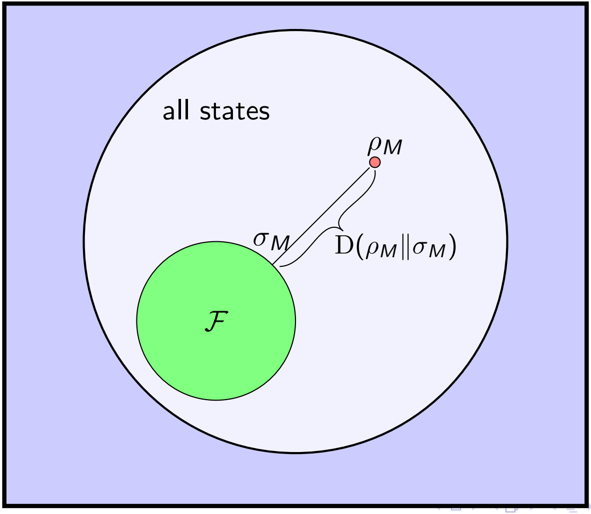
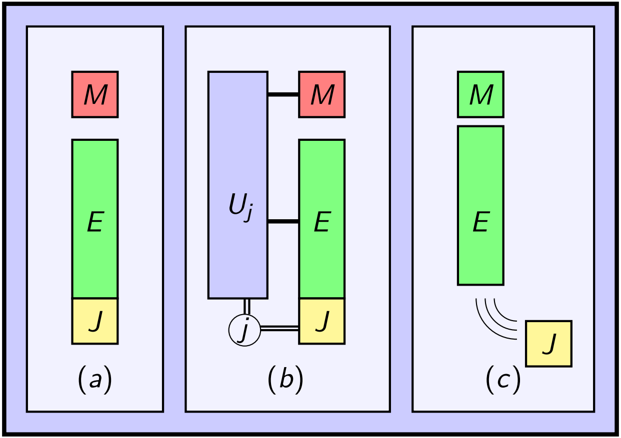
- Max-relative entropy of resource:

$$E_{\varepsilon}^{one-shot}(\rho) = \inf_{\sigma \in \mathcal{F}} \mathrm{D}^{\varepsilon}_{\max}(\rho \| \sigma)$$

Relative entropy of resource: geometric view



Our task



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The role of relative entropy of resource

- Brandao and Gour [Phys. Rev. Lett., 2015] show that in the asymptotic and i.i.d. setting, the rate of transformation of ρ to σ is
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$$\frac{E(\rho)}{E(\sigma)}.$$

- Suggests that $E(\rho)$ is a natural measure for quantifying the amount of resource in a state.

Resource theories in our framework

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- Asymmetry.
  - Wakakuwa [Phys. Rev. A., 2017], where relative entropy of frameness is non-zero. Does not apply to the formulation in Gour, Marvian, Spekkens [Phys. Rev. A., 2009].

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- Quantum thermodynamics.
  - Brandao et. al. [Phys. Rev. Lett., 2013]; Brandao et. al. [PNAS, 2015]; Horodecki, Oppenheim [Nat. Comm., 2013]; Faist et. al. [Nat. Comm., 2015]; Gour et. al. [Phys. Rep. 2015]; Narasimhachar, Gour [Nat. Comm., 2015].

Outline for section 5

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- The converse proofs follow from basic one-shot entropic inequalities adapted to respective settings.

Convex-split Lemma

- A., Devabathini, Jain [Phys. Rev. Lett. 2017, arXiv 2014].
- Let Ψ_{RB}, σ_B be quantum states, $k = \mathrm{D}_{\max}(\Psi_{RB} \| \Psi_R \otimes \sigma_B)$.

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- Let Ψ_{RB}, σ_B be quantum states, $k = \mathrm{D}_{\max}(\Psi_{RB} \| \Psi_R \otimes \sigma_B)$.
- Consider the following quantum state

$$\tau_{RB_1 B_2 \ldots B_N} = \frac{1}{N} \sum_{j=1}^{N} \Psi_{RB_j} \otimes \sigma_{B_1} \otimes \sigma_{B_2} \ldots \otimes \sigma_{B_{j-1}} \otimes \sigma_{B_{j+1}} \ldots \otimes \sigma_{B_N}$$

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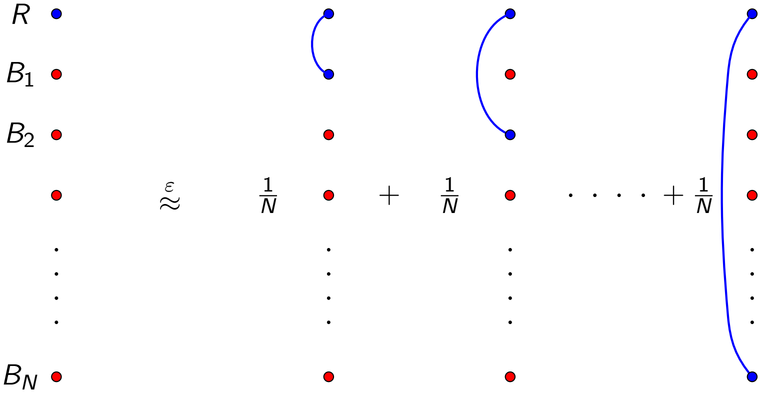
- Then,

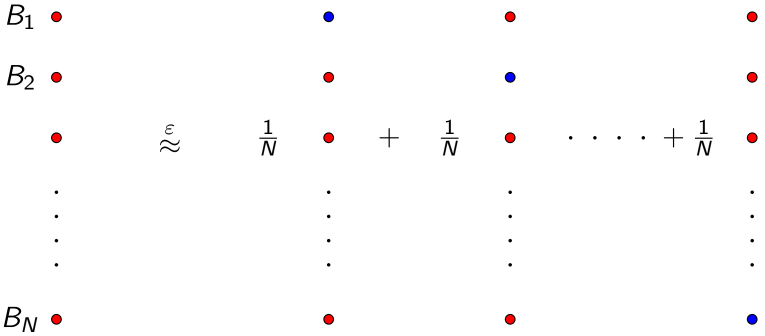
$$\mathrm{D}(\tau_{RB_1B_2\ldots B_N} \| \Psi_R \otimes \sigma_{B_1} \otimes \sigma_{B_2} \ldots \otimes \sigma_{B_N}) \leq \frac{2^k}{N}.$$

Convex-split Lemma: In pictures



If $\log N \geq \mathrm{D}_{\max}(\Psi_{RB} \| \Psi_R \otimes \sigma_B) + \log \frac{1}{\varepsilon}$.

Convex-split Lemma: for our application



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Protocol

ρ_M

σ_{M_1}

σ_{M_2}

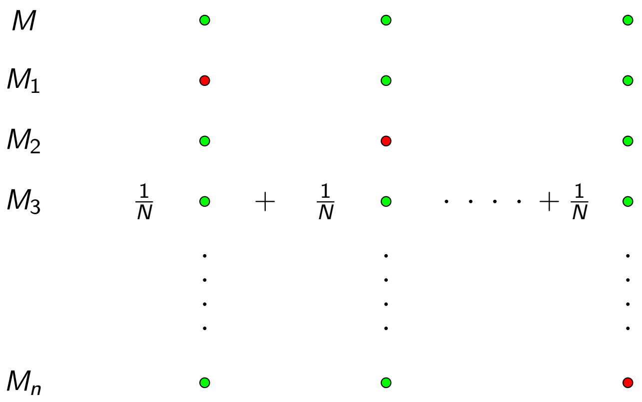
$\vdots$

σ_{M_n}

σ is the state that minimizes

$$E_{\varepsilon}^{one-shot}(\rho) = \inf_{\sigma\in\mathcal{F}} \mathrm{D}_{\max}^{\varepsilon}(\rho\|\sigma).$$

Protocol



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- The achievability protocol uses simple random swap operation.
- Obtains near optimal bounds in settings we consider.
- Open: reduce the amount of catalyst required (compare randomness extraction).
- Open: one shot analogue for resource transformation (Brandao, Gour [Phys. Rev. Lett., 2015]).

Thank you for your attention!