Topological classification of 1D symmetric quantum walks

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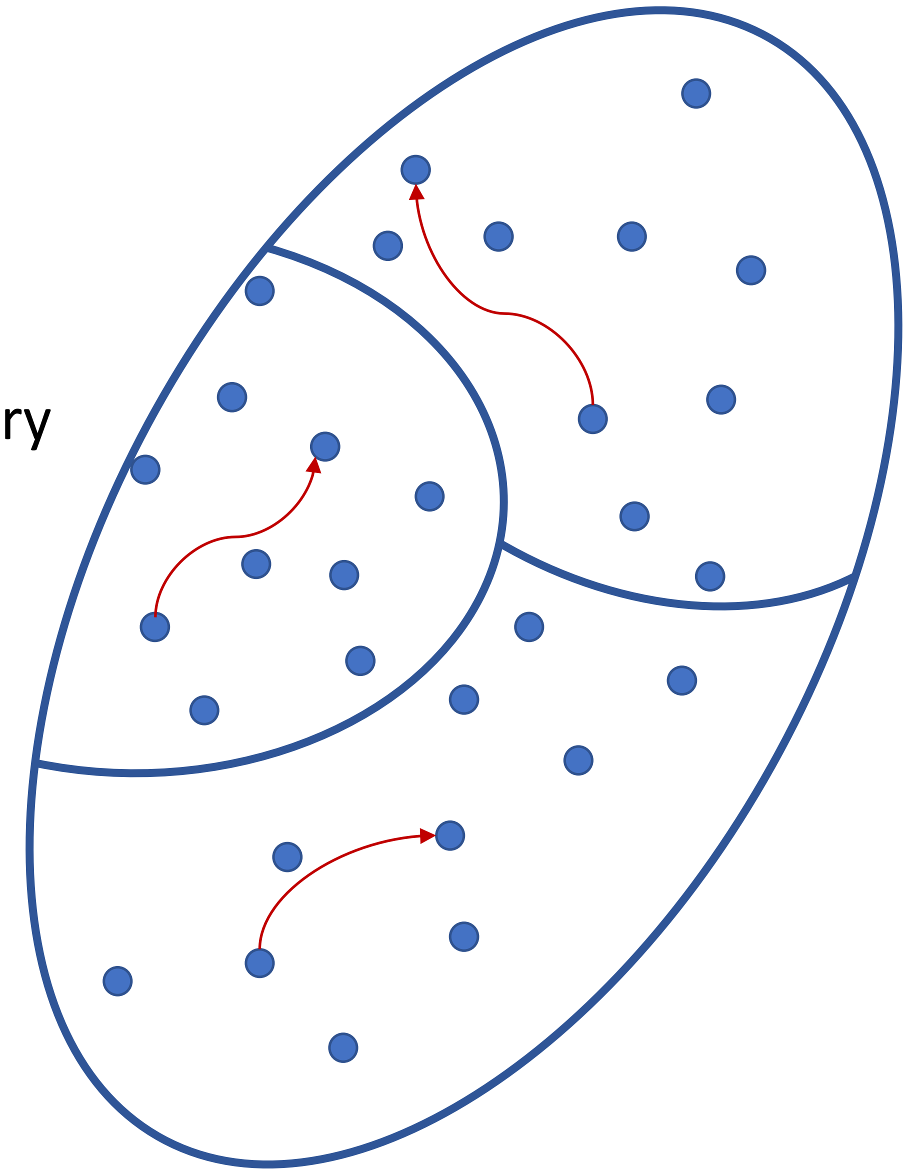
Phase classification

Classification task:

- set of systems to classify
  - Hamiltonians , unitary operators
  - further constraints: spectral gap, locality, symmetry
- allowed operations
  - continuous deformations
  - local perturbations



Classification:

- Is the classification non-trivial?
- Which properties distinguish the phases?
- Bulk-boundary correspondence?

Different flavours of topological order

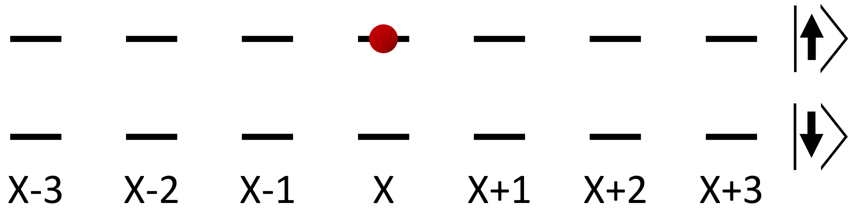
- Gaped local Hamiltonians (free/interacting)
  - only one phase in 1D
  - non-trivial classification in higher dimension
- Gaped local Hamilontians (free/interacting) with symmetry constraints (ten-fold way)
  - non-trivial classification also for 1D
- Floquet systems (time-periodic Hamiltonians) with symmetry
  - non-trivial classification also for 1D
- **Here**: Quantum walks (QWs)

Kitaev (2009); Bravyi et al (2010); Schuch et al (2011); *Affleck et al. (1987);* Zirnbauer (1998);
Altland and Zirnbauer (1997); Kitagawa et al. (2010); Potter et al. (2016)

1. **QWs in a nutshell**
2. Topological classification
3. Completeness of the invariants

QWs in a Nutshell

- discrete time unitary evolution of a single particle on a lattice
- with internal degree of freedom
- strictly local



simple class of examples: Shift-Coin-QWs

$$W = S \cdot C$$

$$\mathcal{H} = \ell_2(\mathbb{Z}) \otimes \mathbb{C}^2$$

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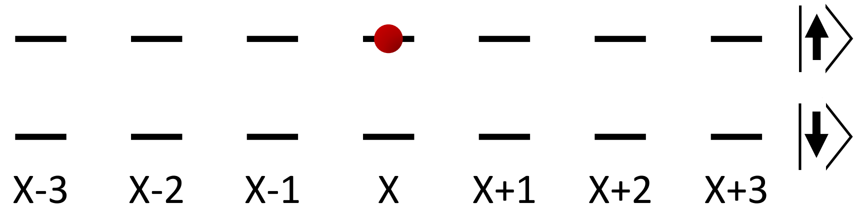

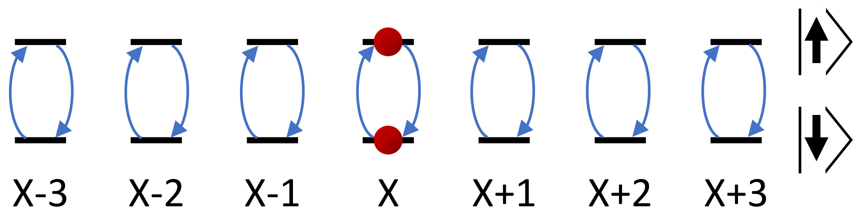

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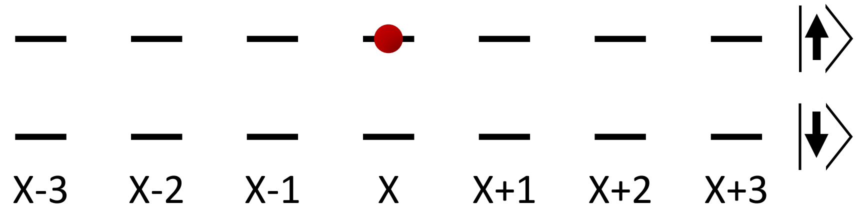
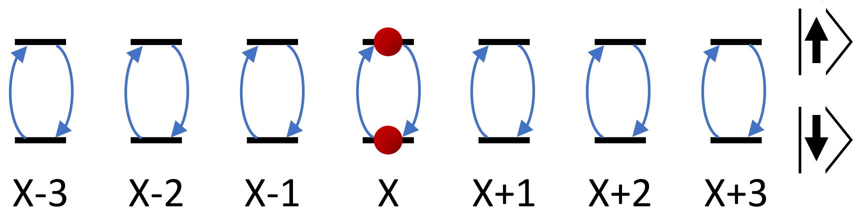
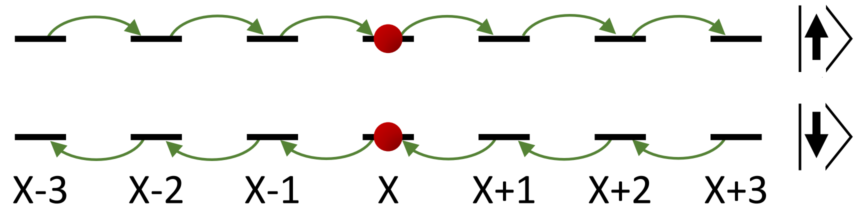
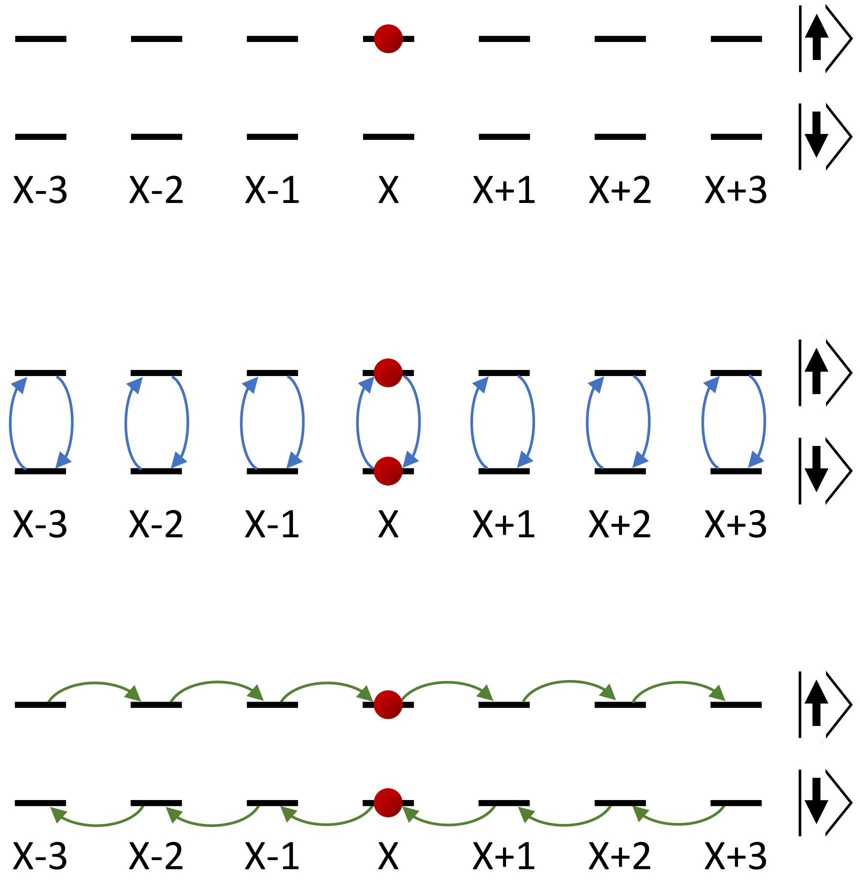






QWs in a Nutshell

- discrete time unitary evolution of a single particle on a lattice
- with internal degree of freedom
- essentially-local: $[P_a, W]$ compact
  - $P_a = \sum_{x \geq a} |x\rangle\langle x| \otimes \mathbb{1}_x$



Quantum Walk (QW):

- essentially local unitary operator

QWs in a Nutshell

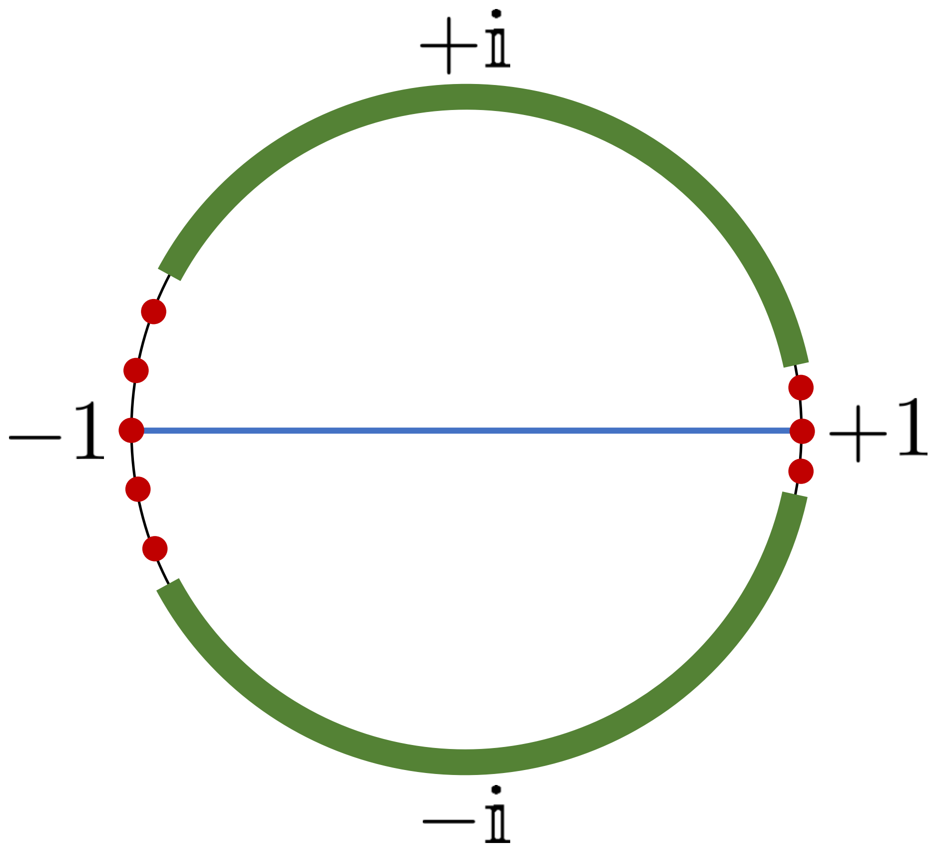
- discrete time unitary evolution of a single particle on a lattice
- with internal degree of freedom
- quasi- local: $[P_a, W]$ compact
  - $\mathcal{H} = \bigoplus_{x\in\mathbb{Z}} \mathcal{H}_x$
  - $P_a = \sum_{x\geq a} |x\rangle\langle x| \otimes \mathbb{1}_x$

Quantum Walk (QW):

- quasi-local unitary operator

Gap condition:

- essentially gaped at ± 1



Motivation for QWs

- single particle time-discrete quantum simulator
  - Exhibits many single particle quantum effects
    - ballistic transport, Anderson localization
    - electric & magnetic fields
- quantum algorithms (Grover search)
- experimental implementations are available (cold atoms, trapped ions, …)
- Index theory for QWs & QCAs
  - non-trivial classification even without symmetries in 1D
  - gives rise to a locally computable invariant: Imbalance of left/right-shift
- Nice explicit examples of symmetric trans. inv. QWs
  - Split-step Walk

Joye (2011); Ahlbrecht et al. (2011); Gross et al. (2012); Kitagawa, Takuya, et al. (2010); Asbóth, and Hideaki (2013)

1. QWs in a nutshell
2. **Topological classification**
3. Completeness of the invariants

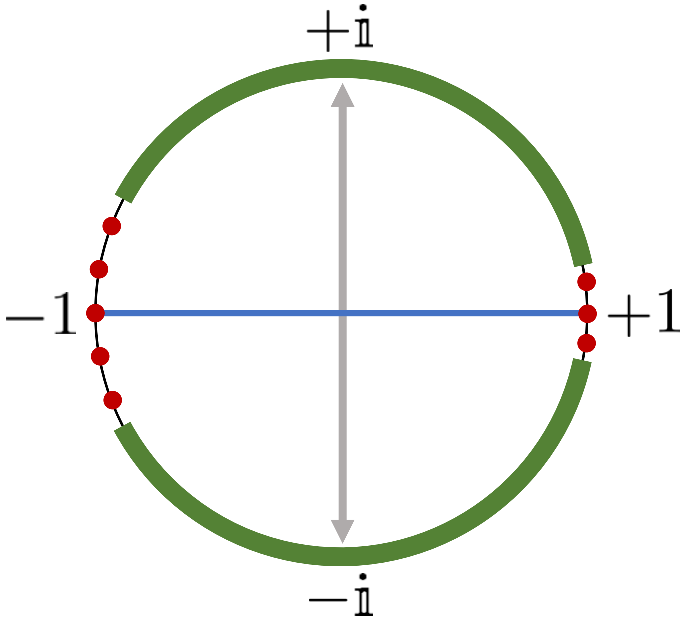
Symmetric QWs

- System $\mathcal{H} = \bigoplus_{x\in\mathbb{Z}} \mathcal{H}_x$
- Symmetry
  - (anti-) unitary operator σ
  - involutive $\sigma^2 = \pm\mathbb{1}$
  - acts „trivally“ on each cell
- 10-fold way

particle-hole	time-reversal	chiral
anit-unitary	anti-unitary	unitary
$W\eta = \eta W$	$W\tau = \tau W^*$	$W\gamma = \gamma W^*$

QW admissible for a subset of $\{\eta, \tau, \gamma\}$

- satisfies all required commutation relations
- is essentially gaped



Zirnbauer (1998); Altland and Zirnbauer (1997)

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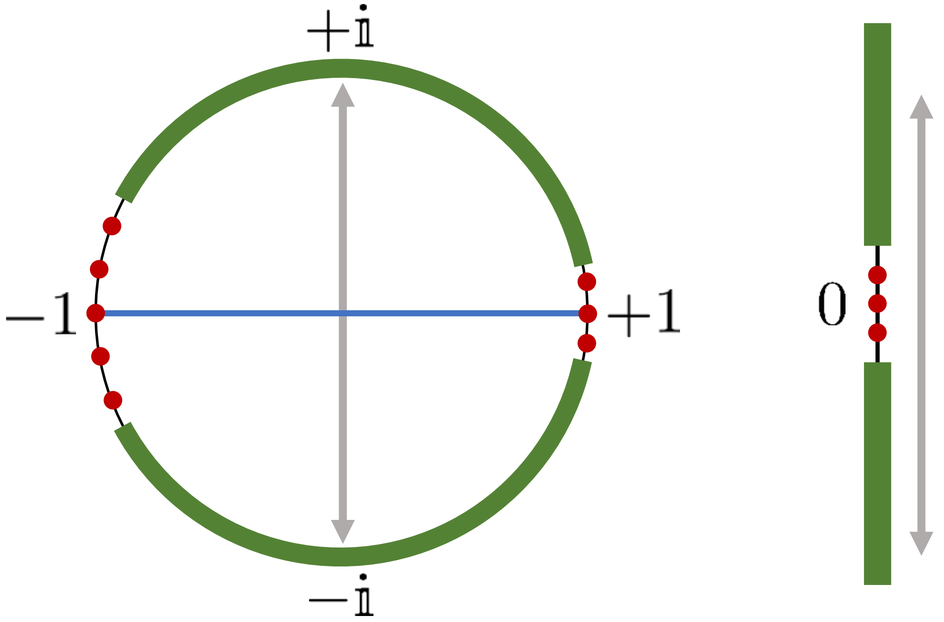

Allowed operations/Perturbations

- System $\mathcal{H} = \bigoplus_{x\in\mathbb{Z}} \mathcal{H}_x$
- W, W' admissible operators
- W' is a
  - **gentle** perturbation if there exists continuous & symmetric path
    $$W \xrightarrow{W_t} W'$$
  - **compact** perturbation if $W - W'$ is compact
  - **local** perturbation if $W - W'$ is local

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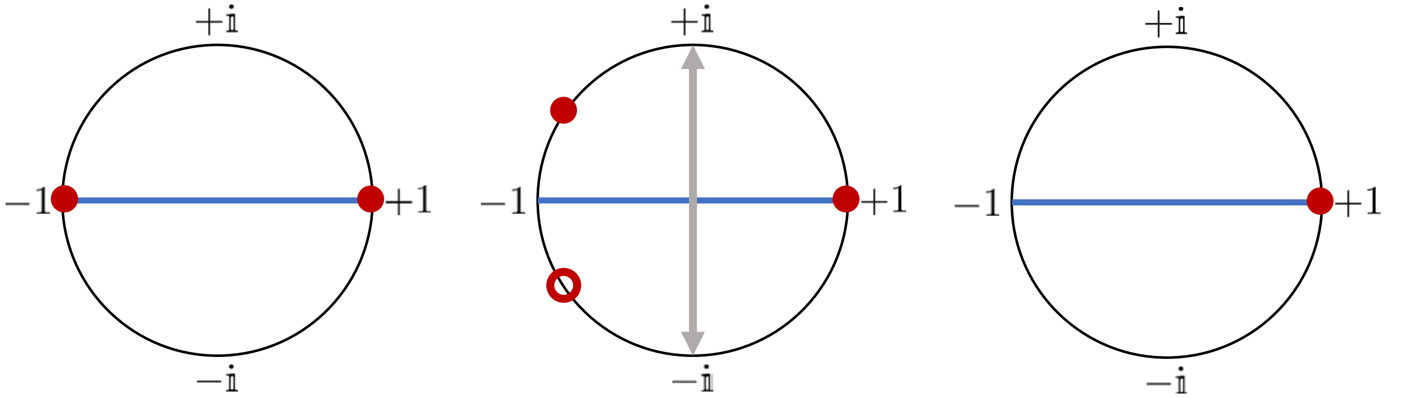
Hamilontians:
all local perturbations are gentle

$$H_t = (1-t)H + tH'$$

Fails for unitary operators

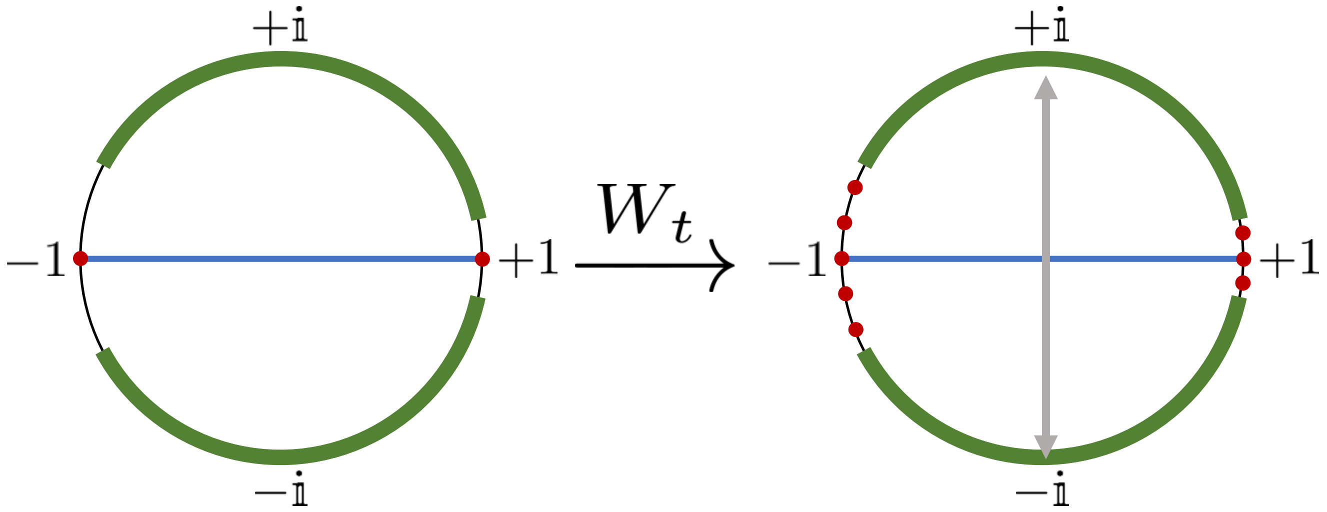
Example: $\mathcal{H} = \mathbb{C}^2$

$$W' = \sigma_z \quad \rightarrow \quad W_t \quad \rightarrow \quad W = \mathbb{1}$$



Intuition from particle-hole symmetry

W admissible for $\eta,\ \eta^2 = \mathbb{1}$



Obervations

- additive under direct sums
- even parity balanced: connected to gaped operator
- reduced problem to finite dimension

Invariant: parity of the dimension of ± 1 eigenspaces

Symmetry index

Example generalizes to 10-fold way

- for any subsets of $\{\eta, \tau, \gamma\}$

  $$\mathrm{si}_\pm : W \mapsto \mathrm{si}_\pm(W) \in \mathbb{Z}_2, \mathbb{Z}$$

- $\mathrm{si}_\pm(W)$ characterizes ± 1 eigenspace
- additive under direct sums
- $\mathrm{si}_\pm(W) = 0$ for balanced eigenspaces
- reduced problem to finite dimension
- independent of spatial structure

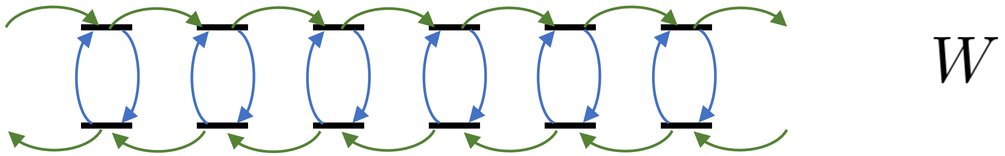
$$\boxed{\mathrm{si}(W) = \mathrm{si}_+(W) + \mathrm{si}_-(W)}$$

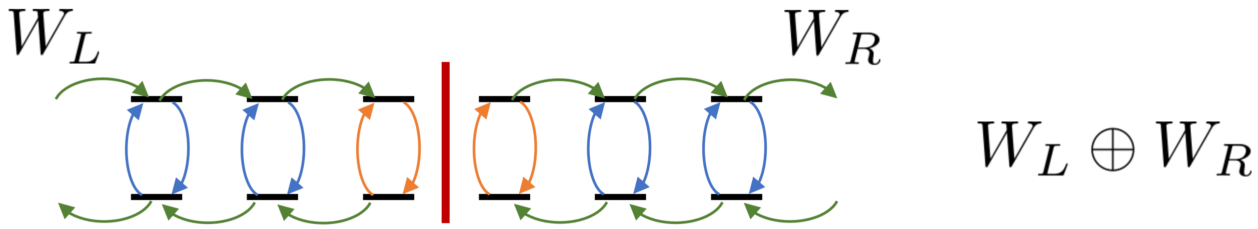
Homotopy invariance

Symmetry index $\mathrm{si}_\pm : W \mapsto \mathrm{si}_\pm(W) \in \mathbb{Z}_2, \mathbb{Z}$

Theorem
Let W be admissible. Then $\exists \varepsilon > 0$ such that
$$\mathrm{si}_\pm(W) = \mathrm{si}_\pm(W')$$
for all admissable W' with $\|W - W'\| < \varepsilon$.

Invariants dependent on spatial structure



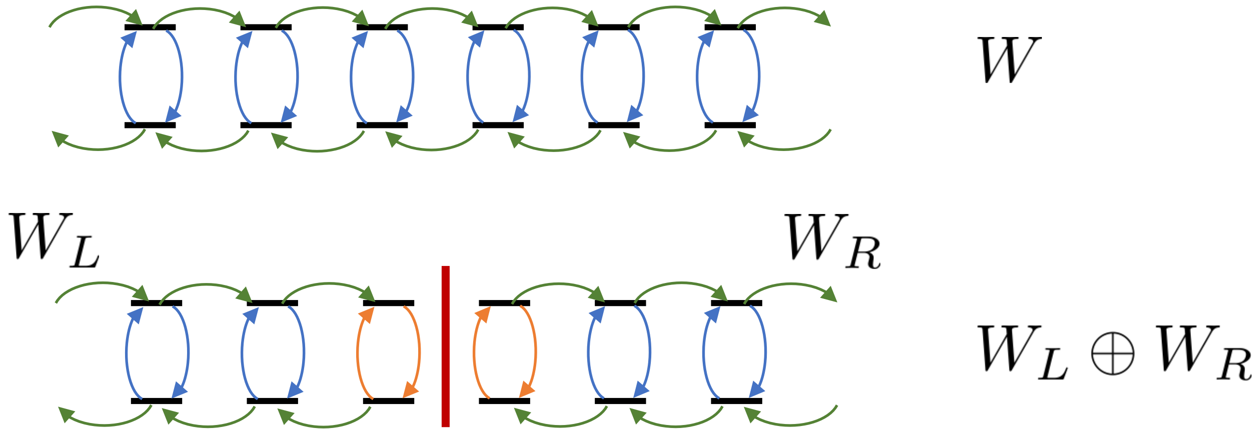


gentle decoupling: $W' = W_L \oplus W_R$ is a gentle perturbation of W

W can be transformed continuously to $W' = W_L \oplus W_R$: $\mathrm{si}_\pm(W) = \mathrm{si}_\pm(W')$

Invariants dependent on spatial structure

Theorem
A gentle decoupling is always possible.



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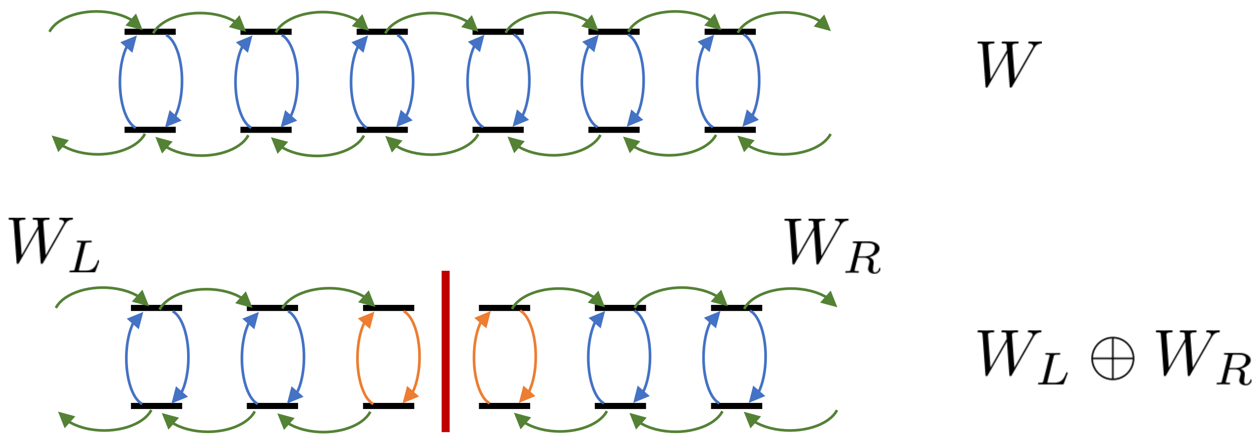


Symmetry index is additive:

$$\mathrm{si}(W) = \mathrm{si}(W') = \mathrm{si}(W_L) + \mathrm{si}(W_R) = \overleftarrow{\mathrm{si}}(W) + \overrightarrow{\mathrm{si}}(W)$$

with: $\overleftarrow{\mathrm{si}}(W) = \mathrm{si}(W_L)$, $\overrightarrow{\mathrm{si}}(W) = \mathrm{si}(W_R)$

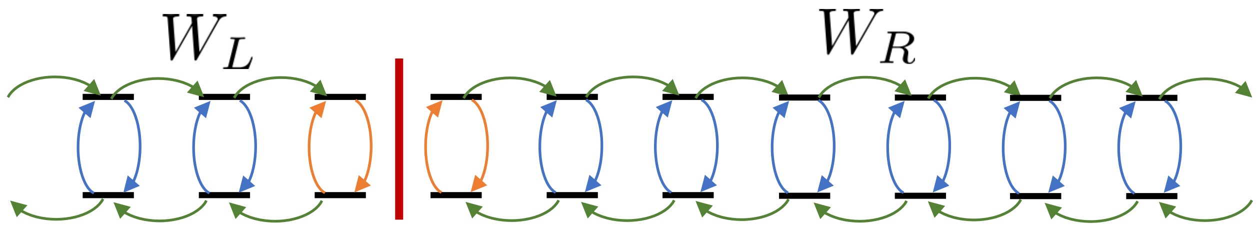
Compact Invariance

Theorem

$\overleftarrow{\mathrm{si}}$ and $\overrightarrow{\mathrm{si}}$ are invariant under **compact** perturbations.

Lemma

$\overleftarrow{\mathrm{si}}$ and $\overrightarrow{\mathrm{si}}$ are independent of the cutting position.



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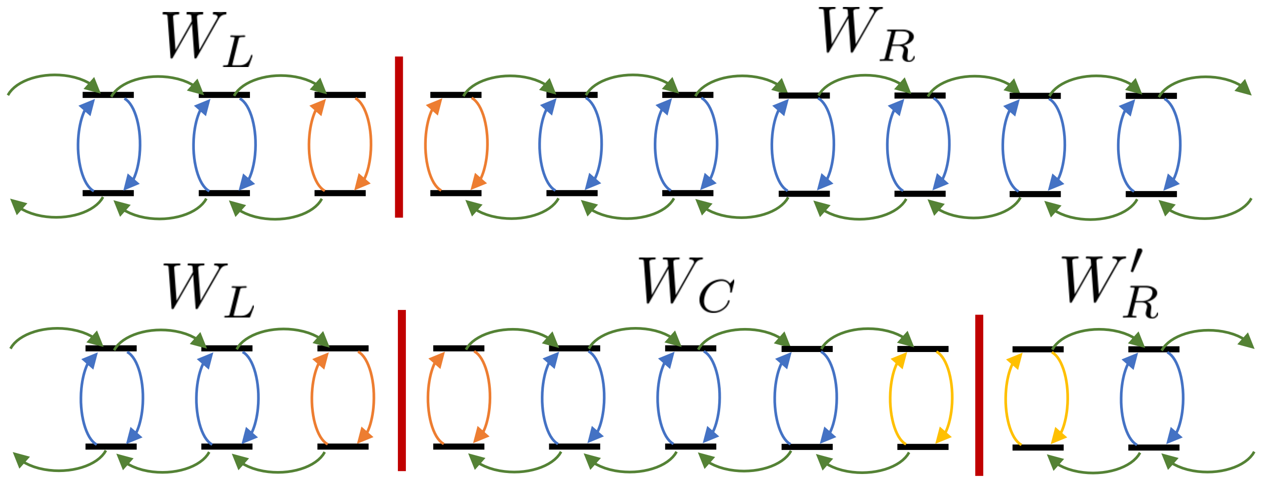
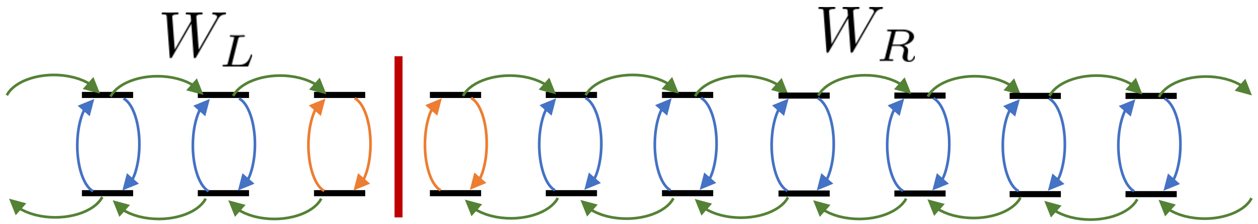


$$\mathrm{si}(W_R) = \mathrm{si}(W_C) + \mathrm{si}(W'_R), \text{ but } \mathrm{si}(W_C) = 0$$

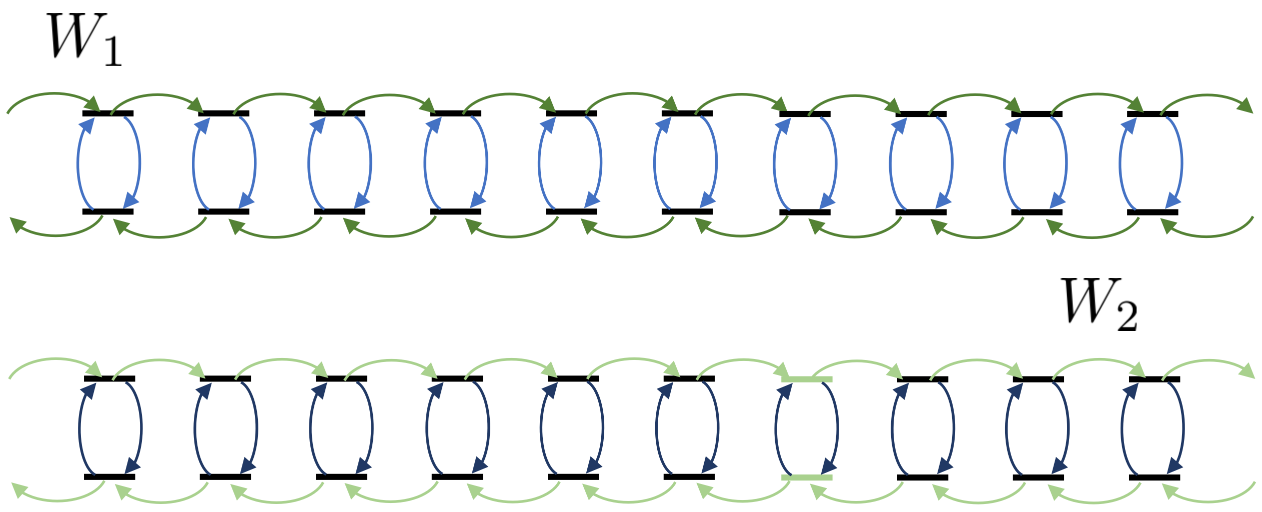
Homotopy vs. Spatial Invariants

	$\mathrm{si}_+(W_L)$	$\mathrm{si}_+(W_R)$	$\mathrm{si}_+(W)$	homotopy stable
	$\mathrm{si}_-(W_L)$	$\mathrm{si}_-(W_R)$	$\mathrm{si}_-(W)$	
stable w.r.t compact pert.	$\overleftarrow{\mathrm{si}}(W)$	$\overrightarrow{\mathrm{si}}(W)$	$si(W)$	



Theorem
Local perturbation W' of W gentle iff $\mathrm{si}_\pm(W) = \mathrm{si}_\pm(W')$.

Bulk-edge correspondence



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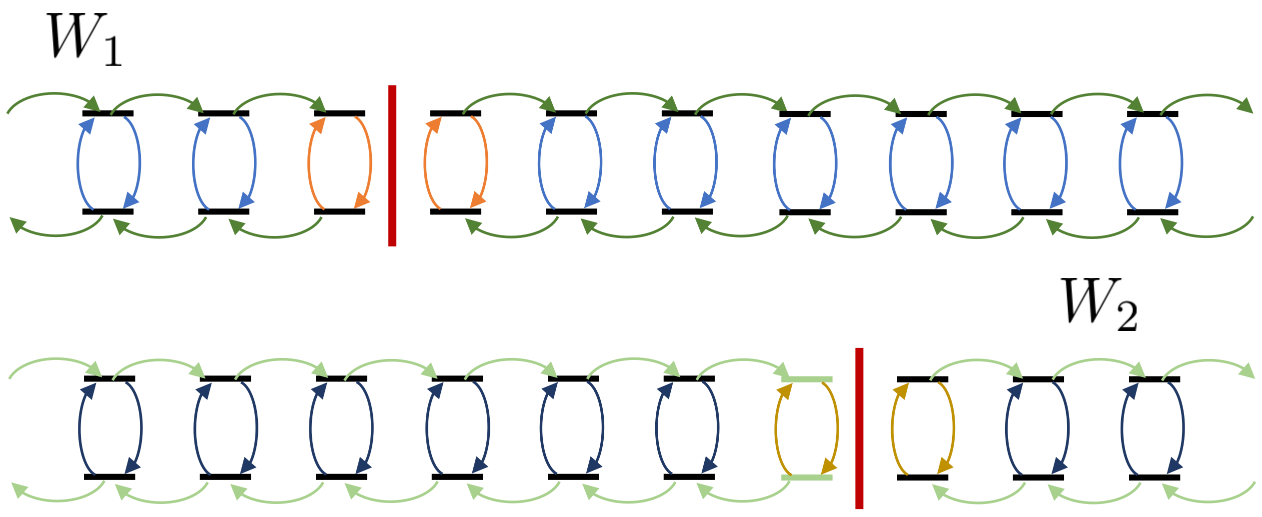
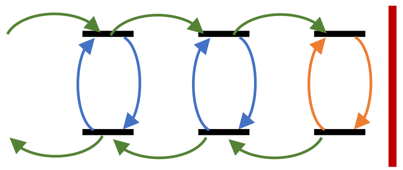
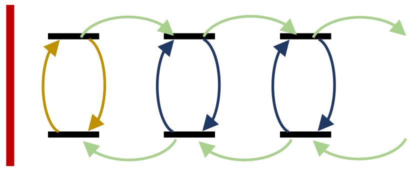


Bulk-edge correspondence

W_1



W_2



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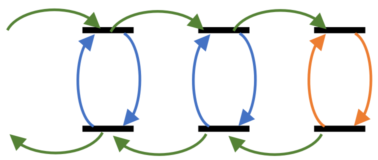
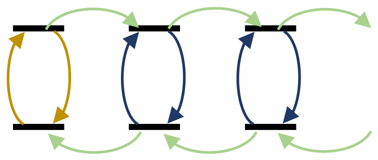


W_2



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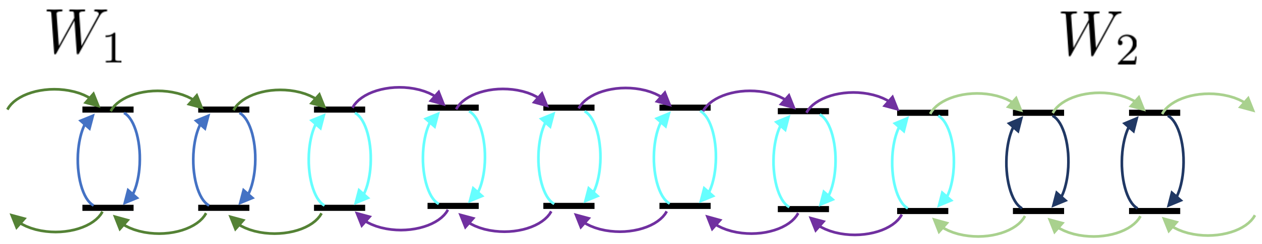

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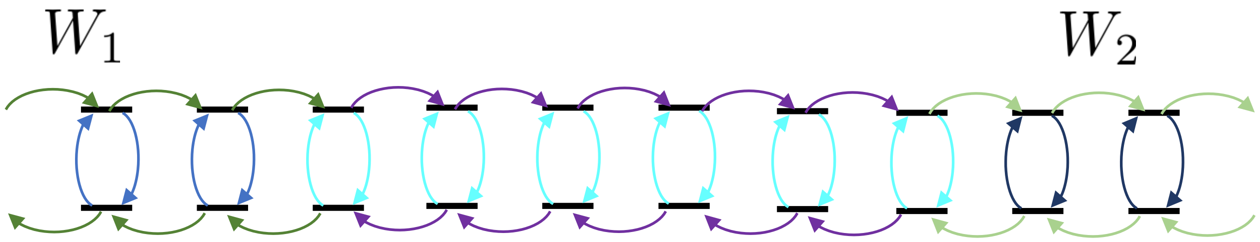
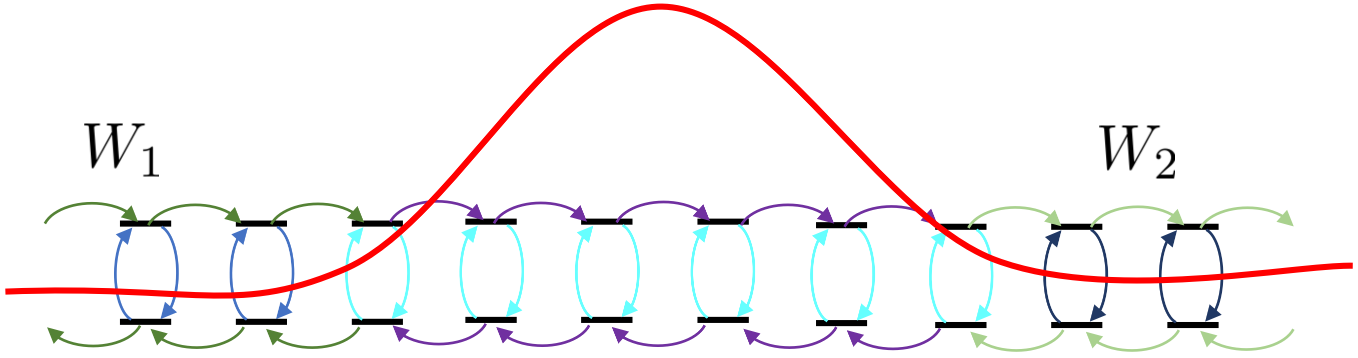


$$\mathrm{si}(W) = \overleftarrow{\mathrm{si}}(W) + \overrightarrow{\mathrm{si}}(W) = -\overrightarrow{\mathrm{si}}(W_1) + \overrightarrow{\mathrm{si}}(W_2)$$

Bulk-edge correspondence



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$$\#(\pm 1 \mathrm{eigenvalues}) \geq |\,\mathrm{si}(W)|$$

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2. Topological classification of symmetric QWs
3. **Completeness of the invariants**

Completeness of the Invariants

Question: Can two walks with the same indices be connected by a symmetric and continuous path?

Three scenarios

(I) quantum walk, gentle perturbations, $\overleftarrow{\mathrm{si}}\,,\ \overrightarrow{\mathrm{si}}\,, \mathrm{si}_-,\ \mathrm{si}_+ = \overleftarrow{\mathrm{si}} + \overrightarrow{\mathrm{si}} - \mathrm{si}_-$

(II) quantum walk, gentle and compact perturbations $\overleftarrow{\mathrm{si}}\,,\ \overrightarrow{\mathrm{si}}$

(III) all essentially local and admissable unitary operators, gentle perturbations $\mathrm{si}_-,\ \mathrm{si}_+$

Completeness of the Invariants

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(I) quantum walk, gentle perturbations, $\overleftarrow{\mathrm{si}}$, $\overrightarrow{\mathrm{si}}$, si_-, $\mathrm{si}_+ = \overleftarrow{\mathrm{si}} + \overrightarrow{\mathrm{si}} - \mathrm{si}_-$

(II) quantum walk, gentle and compact perturbations $\overleftarrow{\mathrm{si}}$, $\overrightarrow{\mathrm{si}}$

(III) all essentially local and admissable unitary operators, gentle perturbations si_-, si_+

Theorem

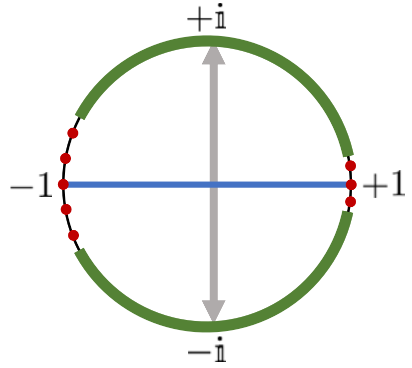
In all three scenarios the invariants are complete for all symmetry types that force the spectrum to be symmetric w.r.t. the real axis

Outlook

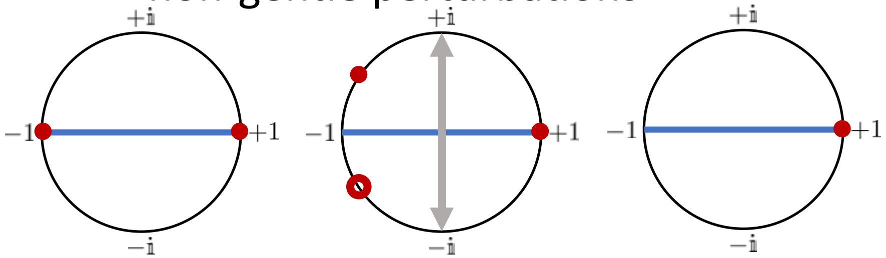
- explicit formulae for invariants in translation invariant setting
- extend construction to higher spatial dimensions
- self-averaging invariants
- extension to quantum cellular automatons

Summary

particle-hole	time-reversal	chiral
anit-unitary	anti-unitary	unitary
$W\eta = \eta W$	$W\tau = \tau W^*$	$W\gamma = \gamma W^*$



non-gentle perturbations



$$
\begin{array}{cc|c}
\mathrm{si}_+(W_L) & \mathrm{si}_+(W_R) & \mathrm{si}_+(W) \\
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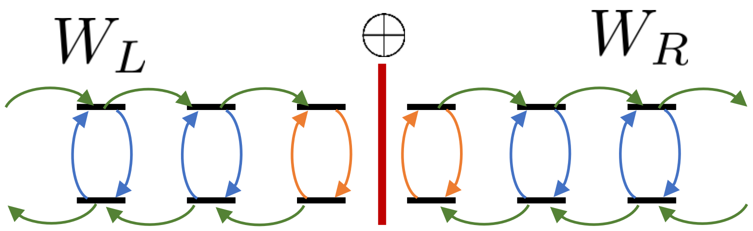
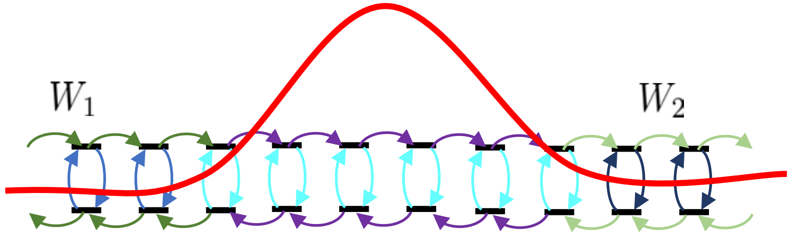
\hline
\overleftarrow{\mathrm{si}}(W) & \overrightarrow{\mathrm{si}}(W) & \mathrm{si}(W)
\end{array}
$$

homotopy stable

compactly stable





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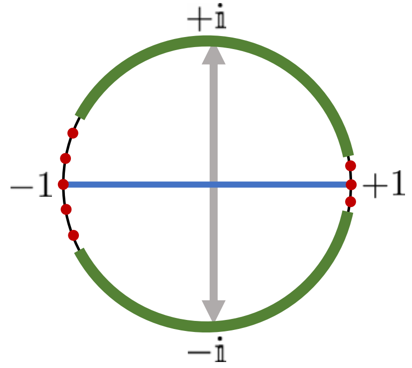

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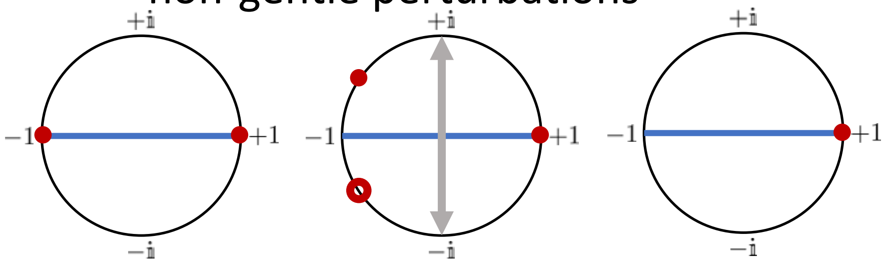

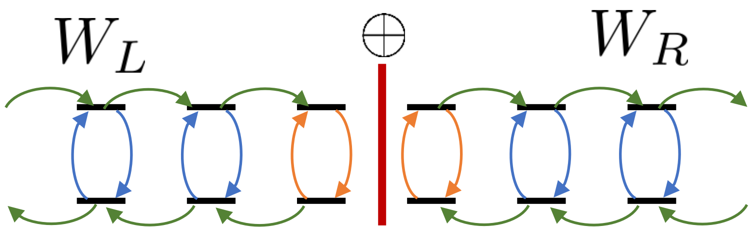

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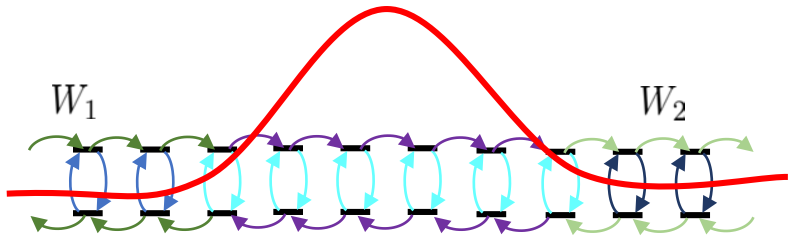
\overleftarrow{\mathrm{si}}(W) & \overrightarrow{\mathrm{si}}(W) & \mathrm{si}(W)
\end{array}\right\} \text{homotopy stable}
$$

$$\underbrace{\overleftarrow{\mathrm{si}}(W) \quad \overrightarrow{\mathrm{si}}(W) \quad \mathrm{si}(W)}_{\text{compactly stable}}$$



Comletness of the invariants in three scenerios

arXiv:1611.04439

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