Estimating the decoherence time using non-commutative Functional Inequalities

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Quantum Information Processing, Monday the 13th of January 2018

Based on Arxiv 1710.01039

Simple definition of the decoherence time

- Consider a quantum system $\mathcal{H} = \mathbb{C}^d$ and denote by $\mathcal{D}_d$ the set of density matrix on $\mathcal{H}$;
- Decoherence is the idea that there exists a **preferred basis** such that the off-diagonal terms of any density matrix disappear in time:

$$\rho = \begin{pmatrix} \rho_1 & & \star \\ & \ddots & \\ \star & & \rho_d \end{pmatrix} \xrightarrow[t \to +\infty]{} \rho_{\text{diag}} := \begin{pmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_d \end{pmatrix} .$$

Define an interpolating family of density matrix between ρ and ρ_{diag} as:

$$\rho_t = e^{-t} \rho + (1 - e^{-t}) \rho_{diag} : \qquad \rho_0 = \rho , \quad \rho_{+\infty} = \rho_{\text{diag}} ;$$

The main topic of this talk is the study of the **decoherence time** ($\|\cdot\|_1 = \mathrm{Tr}|\cdot|$)

$$t(\varepsilon) = \inf \left\{ t \geq 0 ; \left\| \rho_t - \rho_{\text{diag}} \right\|_1 \leq \varepsilon \quad \forall \rho \in \mathcal{D}_d \right\} .$$

Remark: ρ_t is a special instance of (quantum) Markovian evolution.

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1. Decoherence for quantum Markov semigroups
2. Functional inequalities for estimating the decoherence time
3. $\mathbb{L}_p$-regularity
4. Conclusion and open questions

Quantum Markov semigroups

Environment Induced Decoherence, Zurek 1982

Decoherence is a **generic** behavior of **open quantum systems** (at least in the Markovian approximation).

- Evolution of open systems in the **Markovian** regime are modeled by semigroups of **quantum channels** (completely positive and trace preserving superoperators), called **quantum Markov semigroups**:

$$\rho_t = \mathcal{P}_t^\dagger(\rho)\,, \qquad \mathcal{P}_{t+s}^\dagger = \mathcal{P}_t^\dagger \mathcal{P}_s^\dagger\,, \qquad \mathrm{Tr}[\mathcal{P}_t^\dagger(\rho)] = \mathrm{Tr}[\rho]\,;$$

- The generator $\mathcal{L}^\dagger$ defined by $\mathcal{P}_t^\dagger = \exp\, t\,\mathcal{L}^\dagger$ is called a **Lindbladian**.
- In the dual **Heisenberg picture**, one is interested in the evolution of **observables** $X \in \mathcal{B}(\mathcal{H})$;
- In this case, the evolution is given by the dual $\mathcal{P}_t$ of $\mathcal{P}_t^\dagger$, which are **unital** operators.

$$\mathrm{Tr}[\rho\,\mathcal{P}_t(X)] = \mathrm{Tr}[\mathcal{P}_t^\dagger(\rho)\,X] \quad \forall \rho, \forall X\,, \qquad \mathcal{P}_t(I_\mathcal{H}) = I_\mathcal{H}\,.$$

We will always assume that $\mathcal{P}$ has a **full-rank invariant density matrix** σ:

$$\sigma > 0, \quad \mathrm{Tr}[\sigma] = \mathrm{Tr}[\mathcal{P}_t^\dagger(\sigma)] = 1 \quad \text{and} \quad \mathrm{Tr}\,[\sigma\ \mathcal{P}_t(X)] = \mathrm{Tr}\,[\sigma\ X] \quad \forall X \in \mathcal{B}(\mathcal{H}), \forall t \geq 0.$$

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Decoherence for quantum Markov semigroups

We call the **Decoherence-Free algebra** the **largest subalgebra** $\mathcal{N}(\mathcal{P})$ on which $(\mathcal{P}_t)_{t\geq 0}$ reduced to a **non-dissipative**-evolution. It means that there exists a one-parameter unitary group $(U_t)_{t\in\mathbb{R}}$ such that

$$\mathcal{P}_t(X) = U_t^* \, X \, U_t \qquad \forall X \in \mathcal{N}(\mathcal{P}), \quad \forall t \geq 0 \, .$$

On $\mathcal{N}(\mathcal{P})$, the evolution is given by the **Schrödinguer Equation for closed systems**.

Theorem (Carbone, Sasso and Umanità (2013))

*Assume there exists a full-rank invariant density matrix σ. Then there exists a **unique conditional expectation** (projection) $E_{\mathcal{N}}$ from $\mathcal{B}(\mathcal{H})$ to $\mathcal{N}(\mathcal{P})$ such that:*

$$Tr[\sigma \, X] = Tr[\sigma \, E_{\mathcal{N}}(X)] \qquad \forall X \in \mathcal{B}(\mathcal{H})$$

and such that:

$$\lim_{t\to+\infty} \mathcal{P}_t \left(X - E_{\mathcal{N}}[X] \right) = 0 \, .$$

Equivalently, denoting by $E_{\mathcal{N}}^{\dagger}$ the dual of $E_{\mathcal{N}}$, we have $E_{\mathcal{N}}^{\dagger}(\sigma) = \sigma$ and:

$$\lim_{t\to 0} \mathcal{P}_t^{\dagger} \left(\rho - E_{\mathcal{N}}^{\dagger}[\rho] \right) = 0 \qquad \forall \rho \in \mathcal{D}_d \, .$$

Intuitively: $\mathcal{P}_t(X)$ interpolates between X and its projection on $\mathcal{N}(\mathcal{P})$.

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Two examples

Primitive QMS:

One particular case is when $\mathcal{N}(\mathcal{P}) = \mathbb{C}\, I_{\mathcal{H}}$. In this case:

- σ is the unique invariant density matrix;
- $E_{\mathcal{N}}[X] = \mathrm{Tr}[\sigma\, X]\, I_{\mathcal{H}}$ for all $X \in \mathcal{B}(\mathcal{H})$ and $E_{\mathcal{N}}^{\dagger}[\rho] = \sigma$ for all $\rho \in \mathcal{D}_d$;
- One has the limits:

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Decoherent QMS:

Consider the case where $\mathcal{N}(\mathcal{P})$ is the algebra of **diagonal operators** in some preferred orthonormal basis and define:

$$\mathcal{L}_{\mathrm{deco}}^{\dagger}(\rho) = E_{\mathcal{N}}^{\dagger}[\rho] - \rho\,.$$

In this case:

- $E_{\mathcal{N}} = E_{\mathcal{N}}^{\dagger}$ is the projection on diagonal operators;
- $\mathcal{P}_t^{\dagger}(\rho) = e^{-\,t}\,\rho + (1 - e^{-\,t})\, E_{\mathcal{N}}^{\dagger}[\rho]$;
- A full-rank invariant density matrix is the maximally-mixed $\frac{I_d}{d}$;
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What does $\mathcal{N}(\mathcal{P})$ looks like in general?

In the general case, $\mathcal{N}(\mathcal{P})$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ so admits the following structure:

- $\mathcal{H}$ can be decomposed as:
$$\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i \otimes \mathbb{C}^{k_i};$$
- $\mathcal{N}(\mathcal{P})$ can be decomposed as:
$$\mathcal{N}(\mathcal{P}) = \bigoplus_{i \in I} \mathcal{B}(\mathcal{H}_i) \otimes I_{k_i};$$
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$$\mathcal{P}_t(X) = U_t^* \, X \, U_t \, ;$$
- The Hilbert spaces $\mathcal{H}_i$ are called **decoherence-free subsystems**.

Quantum passive error correction:

Decoherence-free subsystems are good candidates for **fault-tolerant universal quantum computation** (Kempe, Bacon, Lidar and Whaley 2000).

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Method

Goal: estimating the decoherence time:

$$t(\varepsilon) = \inf\left\{ t \geq 0 \,;\, \left\| \mathcal{P}_t^\dagger \left(\rho - E_\mathcal{N}^\dagger[\rho] \right) \right\|_1 \leq \varepsilon \quad \forall \rho \in \mathcal{D}_d \right\}.$$

Method: upper bounding with more tracktable functionals:

(Diaconis and Saloff-Coste 1996, Kastoryano and Temme 2013)

$$\left\| \mathcal{P}_t^\dagger \left(\rho - E_\mathcal{N}^\dagger[\rho] \right) \right\|_1 \leq \begin{cases} \sqrt{\chi^2 \left(\mathcal{P}_t^\dagger(\rho) \,,\, \mathcal{P}_t^\dagger \left(E_\mathcal{N}^\dagger[\rho] \right) \right)}\,, \\ \sqrt{2\, D\left(\mathcal{P}_t^\dagger(\rho) \,||\, \mathcal{P}_t^\dagger \left(E_\mathcal{N}^\dagger[\rho] \right) \right)}\,, \end{cases}$$

- where $\chi^2(\rho, \sigma)$ is the χ^2 **divergence** and the bound was derived in (Ruskai 1994):

$$\chi^2(\rho\,,\,\sigma) = \mathrm{Tr}\left[\sigma^{-\frac{1}{2}}\,(\rho - \sigma)\,\sigma^{-\frac{1}{2}}\,(\rho - \sigma)\right] \leq 1/\sigma_{\min}\,,$$

- where $D\,(\rho\,||\,\sigma)$ is the **relative entropy** and the bound is the Pinsker inequality:

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The weight-L_p spaces

The reference density matrix

Define the following density matrix:

$$\sigma_{\mathbf{Tr}} = E_{\mathcal{N}}^{\dagger}\left[\frac{I_{\mathcal{H}}}{d}\right],$$

It is a **full-rank invariant** density matrix (with additional desirable properties...).

- The L^2 scalar product with respect to $\sigma_{\mathbf{Tr}}$ is defined for all $X, Y \in \mathcal{B}(\mathcal{H})$ by:

$$\langle X, Y\rangle_{\sigma_{\mathbf{Tr}}} = \mathrm{Tr}\left[{\sigma_{\mathbf{Tr}}}^{1/2} X^* {\sigma_{\mathbf{Tr}}}^{1/2} Y\right] ;$$

- We consider an **interpolating family** of L^p-norms $\|\cdot\|_{p,\sigma_{\mathbf{Tr}}}$ on $\mathcal{B}(\mathcal{H})$ defined by:

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- A natural map between these spaces is given by, for $1 \leq p, q \leq +\infty$:

$$I_{q,p}(X) = {\sigma_{\mathbf{Tr}}}^{-\frac{1}{2q}}\left({\sigma_{\mathbf{Tr}}}^{\frac{1}{2p}} X\, {\sigma_{\mathbf{Tr}}}^{\frac{1}{2p}}\right)^{\frac{p}{q}} {\sigma_{\mathbf{Tr}}}^{-\frac{1}{2q}} \; : \quad \|X\|_{q,\sigma_{\mathbf{Tr}}}^q = \|I_{q,p}(X)\|_{p,\sigma_{\mathbf{Tr}}}^p .$$

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The case of the χ^2-divergence

- We write $\hat{\mathcal{P}}$ the dual of $\mathcal{P}$ for the above scalar product. It describes the evolution of the **relative density** $\sigma_{\mathrm{Tr}}^{-\frac{1}{2}}\,\rho\,\sigma_{\mathrm{Tr}}^{-\frac{1}{2}}$:

$$\hat{\mathcal{P}}_t\left(\sigma_{\mathrm{Tr}}^{-\frac{1}{2}}\,\rho\,\sigma_{\mathrm{Tr}}^{-\frac{1}{2}}\right) = \sigma_{\mathrm{Tr}}^{-\frac{1}{2}}\,\mathcal{P}_t^{\dagger}(\rho)\,\sigma_{\mathrm{Tr}}^{-\frac{1}{2}}\,;$$

- In terms of the relative density $X = \sigma_{\mathrm{Tr}}^{-\frac{1}{2}}\,\rho\,\sigma_{\mathrm{Tr}}^{-\frac{1}{2}}$, the χ^2-divergence now reads:

$$\chi^2(\rho\,,\,E_{\mathcal{N}}^{\dagger}[\rho]) = \|X - E_{\mathcal{N}}[X]\|_{2,\sigma_{\mathrm{Tr}}}^2\,.$$

- Differentiating the χ^2-divergence leads to

$$\frac{\partial}{\partial t}\bigg|_{t=0}\left\|\hat{\mathcal{P}}_t\left(X - E_{\mathcal{N}}[X]\right)\right\|_{2,\sigma_{\mathrm{Tr}}}^2 = 2\,\langle X, \mathcal{L}(X)\rangle_{\sigma_{\mathrm{Tr}}} \leq 0\,.$$

Exponential decay in L^2-norm

Definition

We define the ***Decoherence-Free Variance*** *(DF-variance) and the* ***Dirichlet form*** *for all $X \in \mathcal{B}(\mathcal{H})$ as*

$$Var_{\mathcal{N}}(X) := \|X - E_{\mathcal{N}}(X)\|^2_{2,\sigma_{\mathbf{Tr}}} \, ,$$
$$\mathcal{E}_{\mathcal{L}}(X) := -\langle X, \mathcal{L}(X)\rangle_{\sigma_{\mathbf{Tr}}} = -\frac{1}{2} \left. \frac{\partial}{\partial t} \right|_{t=0} \left\| \hat{\mathcal{P}}_t \left(X - E_{\mathcal{N}}[X]\right) \right\|^2_{2,\sigma_{\mathbf{Tr}}} .$$

Theorem

Define the ***Decoherence-Free Poincaré Inequality*** *as the existence of a $\lambda \geq 0$ such that for all $X \in \mathcal{B}(\mathcal{H})$ with $X = X^*$:*

$$\lambda \, Var_{\mathcal{N}}(X) \leq \mathcal{E}_{\mathcal{L}}(X) \, .$$

Then one has an ***exponential speed of decoherence*** *in terms of the DF-variance:*

$$Var_{\mathcal{N}}(\mathcal{P}_t(X)) \leq \; Var_{\mathcal{N}}(X) \, e^{-2\lambda t} \qquad \forall X \in \mathcal{B}(\mathcal{H}), \, X = X^* \, .$$

Let $\lambda(\mathcal{L})$ be the best constant in this inequality:

$$\lambda(\mathcal{L}) = \min_{\substack{X \in \mathcal{B}(\mathcal{H}), \\ X = X^*}} \frac{\mathcal{E}_{\mathcal{L}}(X)}{Var_{\mathcal{N}}(X)} \, .$$

Exponential decay in L^2-norm

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Exponential decay in relative entropy

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*We define the **Decoherence-Free relative entropy** (DF-relative entropy) and the **entropy production** for all density matrices $\rho \in \mathcal{D}_d$ as*

$$\mathrm{D}\,(\,\rho\,,\,\mathcal{N}\,) := \mathrm{D}\left(\,\rho\,||\,E^{\dagger}_{\mathcal{N}}[\rho]\,\right)\,,$$

$$EP_{\mathcal{L}}(X) := -\left.\frac{\partial}{\partial t}\right|_{t=0} \mathrm{D}\left(\,\mathcal{P}^{\dagger}_{t}(\rho)\,,\,\mathcal{N}\,\right) = -Tr\big[\mathcal{L}^{\dagger}(\rho)\,(\log\rho - \log\sigma_{Tr})\,\big]\,.$$

Theorem

*Define the **Decoherence-Free modified log-Sobolev Inequality** as the existence of $\alpha \geq 0$ such that for all states $\rho \in \mathcal{D}_d$,*

$$2\alpha\,\mathrm{D}\,(\,\rho\,,\,\mathcal{N}\,) \leq EP_{\mathcal{L}}(\rho)\,.$$

Then, for all $\rho \in \mathcal{D}_d$,

$$\mathrm{D}\left(\,\mathcal{P}^{\dagger}_{t}(\rho)\,,\,\mathcal{N}\,\right) \leq \mathrm{D}\,(\,\rho\,,\,\mathcal{N}\,)\,e^{-2\,\alpha\,t} \qquad \text{for all } t \geq 0\,.$$

Let $\alpha_{\mathcal{N}}(\mathcal{L})$ be the best constant in the previous inequality:

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The decoherence-time of the decoherent QMS

Theorem (Comparison between the two constants)

*Let $\mathcal{P}$ be a **reversible** QMS on $\mathcal{B}(\mathcal{H})$, with generator $\mathcal{L}$. Then the DF-log-Sobolev constant $\alpha_{\mathcal{N}}(\mathcal{L})$ and the spectral gap $\lambda(\mathcal{L})$ satisfy:*

$$\alpha_{\mathcal{N}}(\mathcal{L}) \leq \lambda(\mathcal{L})\,.$$

The decoherent QMS

Recall the definition of the decoherent QMS:

$$\mathcal{P}_t^\dagger(\rho) = e^{-\,t}\,\rho + (1 - e^{-\,t})\,E_{\mathcal{N}}^\dagger[\rho]\,,$$

$$\mathcal{L}_{\text{deco}}(\rho) = E_{\mathcal{N}}^\dagger[\rho] - \rho\,.$$

One has

$$\frac{1}{2} \leq \alpha_{\mathcal{N}}(\mathcal{L}_{\text{deco}}) \leq 1 = \lambda(\mathcal{L}_{\text{deco}});$$

We obtained the following estimates on the decoherence time:

$$t_{\chi^2}(\varepsilon) = \mathcal{O}(\log d)\,,$$

$$t_{LS}(\varepsilon) = \mathcal{O}(\log\log d)\,.$$

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$\mathbb{L}_p$-regularity of the Dirichlet form

Comparison between the Dirichlet form and the entropy production:

The QMS is called (strongly) $\mathbb{L}_1$-*regular* if for all $\rho \in \mathcal{D}_d$,

$$\mathrm{EP}_{\mathcal{L}}(\rho) \geq 4\,\mathcal{E}_{\mathcal{L}}\left(I_{2,1}(\rho)\right) = 4\,\mathcal{E}_{\mathcal{L}}\left(\sigma_{\mathrm{Tr}}^{-\frac{1}{4}}\,\rho^{\frac{1}{2}}\,\sigma_{\mathrm{Tr}}^{-\frac{1}{4}}\right).$$

- This property is **always true** for (reversible) classical Markov semigroups;
- It was proved in certain particular cases for quantum Markov semigroups: unital AND trace-preserving QMS, Davies QMS and or depolarizing QMS.

Theorem

Assume that the QMS $\mathcal{P}$ statisfies the following strong form of ***Detailed Balance Condition****:*

$$Tr[\sigma_{Tr}\,\mathcal{P}_t(X)\,Y] = Tr[\sigma_{Tr}\,X\,\mathcal{P}_t(Y)] \qquad \forall X, Y \in \mathcal{B}(\mathcal{H}).$$

Then $\mathcal{P}$ is (strongly) $\mathbb{L}_p$-regular for all $p \geq 1$ $(1 = \frac{1}{p} + \frac{1}{p'})$:

$$\mathcal{E}_{\mathcal{L}}(I_{2,p}(X)) \leq \frac{p}{2}\,\mathcal{E}_{p,\mathcal{L}}(X) := -\frac{p^2}{4(p-1)}\,\langle I_{p',p}(X), X\rangle_{\sigma_{Tr}} \qquad \forall X \geq 0.$$

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1. Decoherence for quantum Markov semigroups
2. Functional inequalities for estimating the decoherence time
3. $\mathbb{L}_p$-regularity
4. Conclusion and open questions

Conclusion

Summary:

- We define two new functional inequalities that are adapted to the study of decoherence introduced by quantum Markov semigroups;
- The DF-modified logarithmic Sobolev Inequality can provides a speed-up of a factor $\log d$ (where d is the dimension of the system);
- We also obtain strong $\mathbb{L}_p$-regularity of the Dirichlet form under a strong form of reversibilty;

Open questions:

- Can we use **"rapid-decoherence"** to prove stability results in passive error corrections schemes (cf Cubitt, Lucia, Michalakis, Perez-Garcia)?
- Except for one particular situation, we are not able to provide estimates of the modified log-Sobolev constant: this motivates the study of **hypercontractivity** for decohering QMS.

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references

Carbone, Sasso, Umanitá (2013): Decoherence for quantum Markov semi-groups on matrix algebras

Cubitt, Lucia, Michalakis, Perez-Garcia (2015):*Stability of local quantum dissipative systems*

Diaconis, Saloff-Coste (1996): *Logarithmic Sobolev inequalities for finite Markov chains*

Kastoryano, Temme (13):*Quantum logarithmic Sobolev inequalities and rapid mixing*

Kempe, Bacon, Lidar, Whaley (2000): *Theory of decoherence-free fault-tolerant universal quantum computation*

Ruskai (1994): *Beyond strong subadditivity? Improved bounds on the contraction of generalized relative entropy*

Zurek (1982): Environment-induced superselection rules