Quantifying decoherence of quantum Markov semigroups via hypercontractivity.

Cambyse Rouzé (University of Cambridge)
Joint work with Dr. Ivan Bardet (IHES)

Quantum Information Processing, Monday, January 15

- Assumption: all the QMS possess a full-rank invariant state ("**decohering**"/"**non-primitive**").

- Recall the **modified logarithmic Sobolev inequality** in the primitive case:

$$\boxed{2\alpha_1\, D(\rho\|\sigma) \leq \mathrm{EP}_{\mathcal{L}}(\rho)} \qquad (\mathrm{MLSI}(\alpha_1))$$

- Can be interpreted as limit $p \to 1$ of a family of functional inequalities called **(strong) p-logarithmic Sobolev inequalities** [Diaconis Saloff Coste 96, Olkiewicz Zegarlinski 99, Kastoryano Temme 13,Temme Pastawski Kastoryano 14,Müller-Hermes Franca Wolf 16,...]: given $X > 0$,

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Logarithmic Sobolev inequalities and hypercontractivity

- An important special case is $\mathrm{sLSI}_2(\alpha_2)$:

$$\boxed{\alpha_2 \,\mathrm{Ent}_2(X) \leq \mathcal{E}_{2,\mathcal{L}}(X) \equiv -\langle X, \mathcal{L}(X)\rangle_\sigma} \qquad (\mathrm{sLSI}_2(\alpha_2))$$

- Recall Ivan's talk, for a QMS satisfying the strong $\mathbb{L}_1$-regularity,

$$\boxed{0 < \alpha_2 \leq \alpha_1}$$

- $\mathrm{sLSI}_2(\alpha_2)$ related to the notion of hypercontractivity of the QMS:

$$\boxed{\forall 1 \leq q \leq p < \infty, \quad \|\mathcal{P}_t(X)\|_{p,\sigma} \leq \|X\|_{q,\sigma}, \qquad t \geq \frac{1}{2\alpha_2}\log\frac{p-1}{q-1}} \qquad (\mathrm{HC}_q(\alpha_2))$$

Theorem (Quantum Gross lemma [Olkiewicz Zegarlinski 99])

If $(\mathcal{P}_t)_{t\geq 0}$ is strongly $\mathbb{L}_p$-regular, then $\mathrm{HC}_q(\alpha_2) \Leftrightarrow \mathrm{sLSI}_2(\alpha_2)$.

sLSI_2 provides an estimate on α_1 from which mixing times can be derived.

One can also derive mixing times directly from HC_2 [Diaconis Saloff-Coste 96].

$\rightarrow$ **Can we extend these concepts to estimate $\alpha_{\mathcal{N}}(\mathcal{L})$/decoherence times?**

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- **Requirements:** we are looking for a family of norms $\|.\|_{(q,p),\,\mathcal{N}}$:
  - reducing to $\|.\|_{p,\sigma}$ for primitive QMS with unique full-rank invariant state σ.
  - so that $\|X\|_{(q,p),\,\mathcal{N}} = \|X\|_{q,\,\sigma_{\mathrm{Tr}}}$ for all $X \in \mathcal{N}(\mathcal{P})$.
  - rendering QMS contractive for all $1 \leq q \leq p \leq \infty$.

- For unital QMS with $\mathcal{N}(\mathcal{P}) := \mathcal{B}(\mathcal{H}_A) \otimes \mathbb{I}_{\mathcal{H}_B}$ [Beigi King 16]: (normalized) Pisier norms [Pisier 93] do the job. For $1 \leq q \leq p$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$:

$$\|X\|_{(q,p),\mathcal{N}} := \inf_{X=AYB} \left\{ \|A\|_{2r,\sigma_{\mathrm{Tr}}} \|B\|_{2r,\sigma_{\mathrm{Tr}}} \|Y\|_{p,\sigma_{\mathrm{Tr}}};\ A, B \in \mathcal{N}(\mathcal{P}),\ Y \in \mathcal{B}(\mathcal{H}) \right\},$$

where $\sigma_{\mathrm{Tr}} := \frac{\mathbb{I}_{AB}}{d_A\, d_B}$. They proved that the quantum Gross lemma can be extended to this case, for an appropriate associated notion of sLSI.

- **Problems**:

  The positivity of the (strong) log-Sobolev constant was not answered.

  Can we find adequate norms to extend this setting to any non-primitive QMS?

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- In [Junge Parcet 10], the authors defined augmented $\mathbb{L}_p$ norms: given a C^*-algebra $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$, $1 \leq q \leq p$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$:

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$$\|Y\|_{(p,q),\,\mathcal{N}} := \sup_{A,\,B \in \mathcal{N}(\mathcal{P})} \frac{\|AYB\|_{q,\,\sigma_{\mathrm{Tr}}}}{\|A\|_{2r,\,\sigma_{\mathrm{Tr}}} \|B\|_{2r,\,\sigma_{\mathrm{Tr}}}}$$

Proposition (Properties of the augmented $\mathbb{L}_p$ norms)

1. *Hölder's inequality:* $|\langle X, Y \rangle_{\sigma_{\mathrm{Tr}}}| \leq \|X\|_{(q,p)\,\mathcal{N}} \, \|Y\|_{(q',p'),\,\mathcal{N}}$
2. *Duality:* $\|X\|_{(q,p),\,\mathcal{N}} = \sup \left\{ |\langle X, Y \rangle_{\sigma_{\mathrm{Tr}}}| :\ \|Y\|_{(q',p'),\,\mathcal{N}} = 1 \right\}$
3. *Reduction to $\mathbb{L}_p(\sigma)$ norms:* $\mathcal{N} = \mathbb{C}\,\mathbb{I} \;\Rightarrow\; \|X\|_{(q,p),\,\mathcal{N}} = \|X\|_{p,\,\sigma_{\mathrm{Tr}}}$
4. *Collapse on $\mathcal{N}$:* $\forall\, X \in \mathcal{N},\ \ \|X\|_{(q,p),\,\mathcal{N}} = \|X\|_{q,\,\sigma_{\mathrm{Tr}}}$
5. *Contractivity: for $\mathcal{N} \equiv \mathcal{N}(\mathcal{P})$:* $\|\mathcal{P}_t(X)\|_{(q,p),\,\mathcal{N}} \leq \|X\|_{(q,p),\,\mathcal{N}}$
6. $\|.\|_{(q,p),\,\mathcal{N}}$ *constitutes an interpolating family of norms.*

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$$\mathcal{E}_{q,\mathcal{L}}(X) := -\frac{q}{2(q-1)} \langle I_{p,q}(X), \mathcal{L}(X) \rangle_{\sigma_{\mathrm{Tr}}} \qquad \textbf{(DF-}\mathbb{L}_q\textbf{ Dirichlet form)}$$

For a decohering QMS $(\mathcal{P}_t)_{t\geq 0}$, define

- **Weak q-DF logarithmic Sobolev inequality**: for any $X > 0$,

$$\boxed{\mathrm{Ent}_{q,\mathcal{N}}(X) \leq c\, \mathcal{E}_{q,\mathcal{L}}(X) + d \|X\|^q_{q,\sigma_{\mathrm{Tr}}}} \qquad (\mathrm{wLSI}_{q,\mathcal{N}}(c,d))$$

- **Weak q-DF hypercontractivity**: for any $1 \leq q \leq p$, any $X \in \mathcal{B}(\mathcal{H})$, and $t \geq \frac{c}{2} \log \left(\frac{p-1}{q-1} \right)$:

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- $\mathrm{wHC}_q(c,0) = \mathrm{HC}_q(\alpha_q)$, $\alpha_q = c^{-1}$.

Theorem (Extension of the quantum Gross lemma to non-primitive QMS)

Let $(\mathcal{P}_t)_{t\geq 0}$ a decohering QMS. Then,

(i) $\mathrm{wHC}_{q,\mathcal{N}}(c,d) \Rightarrow \mathrm{wLSI}_{q,\mathcal{N}}(c,d)$ *for all* $q \geq 1$.

(ii) *In the case when* $\mathcal{N}(\mathcal{P}) = \mathcal{B}(\mathcal{H}_A) \otimes \mathbb{I}_{\mathcal{H}_B}$, *and* $(\mathcal{P}_t)_{t\geq 0}$ *satisfies strong* $\mathbb{L}_p$ *regularity,* $\mathrm{wLSI}_{2,\mathcal{N}}(c,d) \Rightarrow \mathrm{wHC}_{q,\mathcal{N}}(c,d)$.

Intuition: $\mathrm{wLSI}_{q,\mathcal{N}}$ is the infinitesimal formulation of $\mathrm{wHC}_{q,\mathcal{N}}$.

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- In the non-primitive case, one can recover a weaker result:

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For any reversible QMS, $\mathrm{wLSI}_{2,\mathcal{N}}\left(\frac{2+\log(\|\sigma_{\mathrm{Tr}}^{-1}\|_\infty)}{2\lambda(\mathcal{L})}, \log\sqrt{2}\right)$ *holds.*

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- $\mathrm{wHC}_{2,\mathcal{N}}(0, \sqrt{\log \|\sigma_{\mathrm{Tr}}^{-1}\|_\infty})$ always holds, but provides rather loose bounds on decoherence times (exact same as Poincaré).
- Since we showed that $\mathrm{wLSI}_{2,\mathcal{N}}(c, \log\sqrt{2})$ always holds for

$$c := \frac{2 + \log \|\sigma_{\mathrm{Tr}}^{-1}\|_\infty}{2\lambda(\mathcal{L})}$$

and $\mathrm{wLSI}_{2,\mathcal{N}}(c, \log\sqrt{2}) \Rightarrow \mathrm{wHC}_{2,\mathcal{N}}(c, \log\sqrt{2})$ in the case $\mathcal{N}(\mathcal{P}) = \mathcal{B}(\mathcal{H}_A) \otimes \mathbb{I}_{\mathcal{H}_B}$. Hence

Corollary

When $\mathcal{N}(\mathcal{P}) = \mathcal{B}(\mathcal{H}_A) \otimes \mathbb{I}_{\mathcal{H}_B}$, and under $\sigma_{\mathrm{Tr}}-$DBC,

$$\boxed{\|\rho_t - \rho_{\mathcal{N}}\|_1 \leq \mathrm{e}^{1+\log\sqrt{2}} (\log \|\sigma_{\mathrm{Tr}}^{-1}\|_\infty)^{\frac{c}{2\lambda(\mathcal{L})}} \, \mathrm{e}^{-\lambda(\mathcal{L})t}}$$

- **Summary**:

  We extended functional analytical tools (HC,LSI) to the case of non-primitive QMS.

  Weak version of $\mathrm{LSI}_{2,\mathcal{N}}$ holds generically in finite dimensions, with universal constants.

  We used $\mathrm{wHC}_{2,\mathcal{N}}$ to derive bounds on the times to decoherence of non-primitive QMS.

  These bounds are tighter than the ones derived from Poincaré's inequality in the case when $\mathcal{N}(\mathcal{P}) = \mathcal{B}(\mathcal{H}_A) \otimes \mathbb{I}_{\mathcal{H}_B}$.

- **Open questions**:

  Finding optimal weak constants $\mathrm{HC}_{2,\mathcal{N}}(c, d)$ depending on $\mathcal{N}(\mathcal{P})$.

  Can we get the converse of Gross' lemma for a general non-primitive QMS?

Thank you for your attention.

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