Measuring Quantum Entropy

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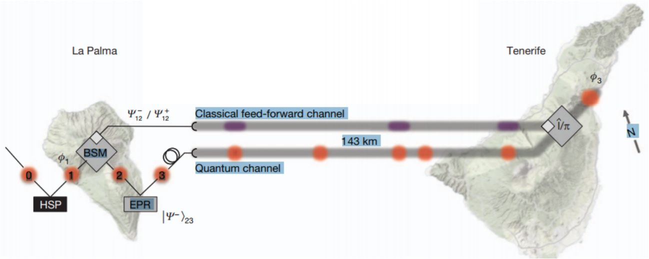
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Quantum Information Processing, January 2018

Motivation



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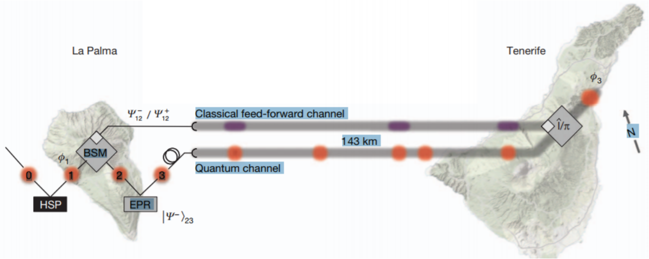

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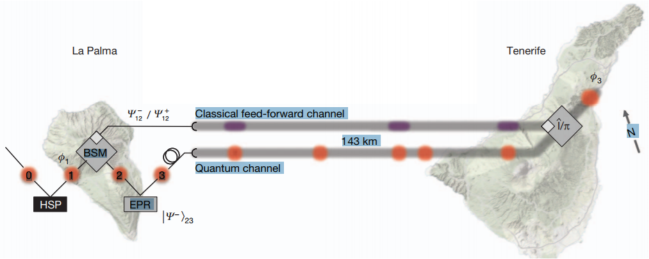

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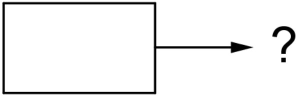

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[Montanaro and de Wolf '13]

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- Testing if $\rho = \sigma$ for some known σ (next talk!)

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Results: Rényi Entropy for Integral α

Theorem (AISW '17)

For $\alpha \in \mathbb{N} \setminus \{1\}$,

$$C(S_\alpha, d, \varepsilon) = \Theta\left(\max\left\{\frac{d^{1-1/\alpha}}{\varepsilon^2}, \frac{d^{2-2/\alpha}}{\varepsilon^{2/\alpha}}\right\}\right)$$

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- Classical estimation

$$C(H_\alpha, d, \varepsilon) = \Theta\left(\frac{d^{1-1/\alpha}}{\varepsilon^2}\right) \quad \text{[AOST15]}$$

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- Classical case [VV11,WY16,JVHW15]

$$C(H, d, \varepsilon) = \Theta\left(\frac{d}{\varepsilon \log d} + \frac{\log^2 d}{\varepsilon^2}\right)$$

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A quantum measurement, called weak Schur sampling, is optimal for estimating unitarily invariant properties [KW01,CHW07,Har05,CHr06].

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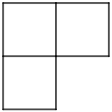

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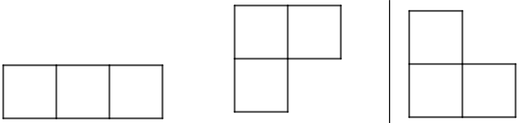

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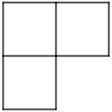

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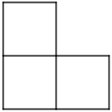

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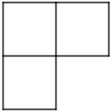

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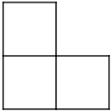

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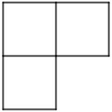

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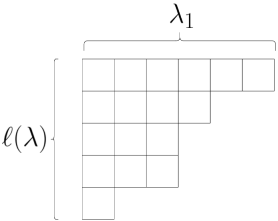

Figure: English Young diagram for the partition $\boldsymbol{\lambda} = (6, 4, 3, 3, 1)$.

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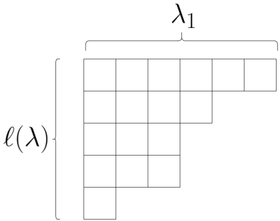

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- Since $\boldsymbol{\eta} - X^n - \boldsymbol{\lambda}(X^n)$ is a Markov chain,

Estimating a quantum state property is at least as hard as estimating the same property in the classical setting

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- Schur polynomial

$$s_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_d) = \sum_{\substack{T\text{:semi-standard tableau} \\ \text{of shape } \boldsymbol{\lambda}}} \prod_{i=1}^{d} x_i^{\{\#\text{ of occurences of } i \text{ in } T\}}$$

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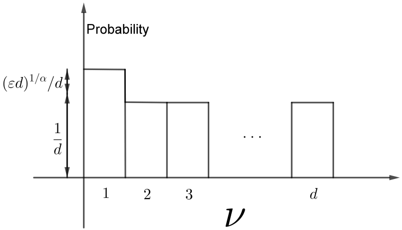

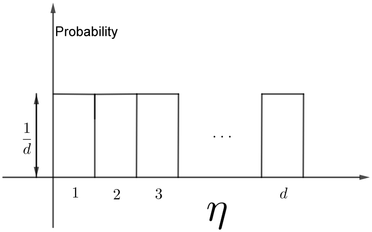

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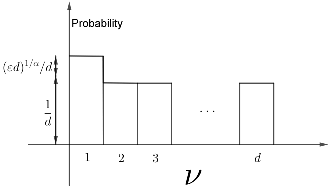

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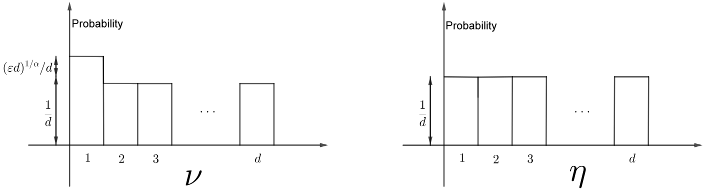

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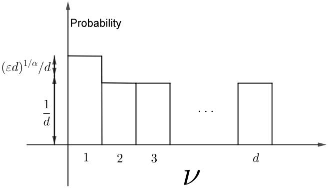

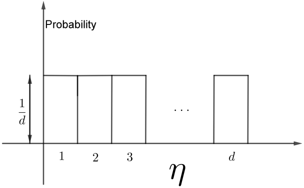

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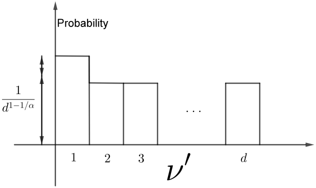

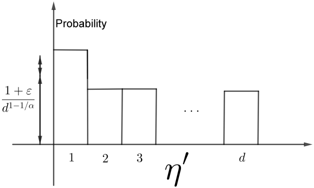

Results: von Neumann Entropy

Theorem (AISW '17)

$$C(S, d, \varepsilon) \leq O\left(\frac{d^2}{\varepsilon^2} + \frac{\log^2(1/\varepsilon)}{\varepsilon^2}\right).$$

Moreover, the empirical estimator (EYD) requires $\Omega\left(\frac{d^2}{\varepsilon}\right)$ *copies to estimate von Neumann entropy.*

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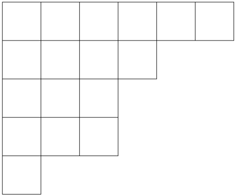

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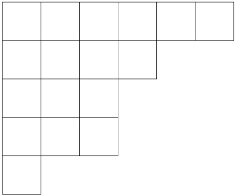

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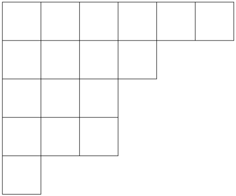

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- It is known that [ARS88,KW01]

$$\mathbf{E}_{SW_\eta}\left[\frac{\boldsymbol{\lambda}}{n}\right] \xrightarrow{n\to\infty} \boldsymbol{\eta}$$

von Neumann Entropy

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Theorem

The empirical entropy estimate satisfies:

$$\mathbf{E}\left[\left(\widehat{S(\boldsymbol{\lambda})} - S(\rho)\right)^2\right] \leq O\left(\frac{d^4}{n^2} + \frac{d^2}{n} + \frac{\log^2 n}{n}\right).$$

Hence,

$$C(S, d, \varepsilon) \leq O\left(\frac{d^2}{\varepsilon^2} + \frac{\log^2(1/\varepsilon)}{\varepsilon^2}\right).$$

von Neumann Entropy - Proof Sketch

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  - Previously the best known lower bound was a constant (0.01) [OW15].

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Results: Non-integral Rényi Entropy

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For $\alpha > 1$,

$$C(S, d, \varepsilon) \leq O\left(\frac{d^2}{\varepsilon^2}\right).$$

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Moreover, the empirical estimator (EYD) requires $\Omega\left(\frac{d^{1+1/\alpha}}{\varepsilon^{1/\alpha}}\right)$ *copies to estimate* $S_\alpha(\rho)$.

Summary

- Integral Rényi entropy with $\alpha > 1$:

$$C(S_\alpha, d, \epsilon) = \Theta\left(\max\left\{\frac{d^{1-1/\alpha}}{\varepsilon^2}, \frac{d^{2-2/\alpha}}{\varepsilon^{2/\alpha}}\right\}\right)$$

- von Neumann and non-integral Rényi entropies:

Table: Copy complexity of empirical estimators

α	Upper Bound	Lower Bound
von Neumann	$O(d^2/\varepsilon^2)$	$\Omega(d^2/\varepsilon)$
$\alpha > 1$	$O(d^2/\varepsilon^2)$	$\Omega(d^2/\varepsilon)$
$\alpha < 1$	$O(d^{2/\alpha}/\varepsilon^{2/\alpha})$	$\Omega(d^{1+1/\alpha}/\varepsilon^{1/\alpha})$

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