# Entanglement requirements for non-local games

arXiv:1703.08618: Slofstra,
The set of quantum correlations is not closed

arXiv:1711.10676: Slofstra & V.,
Entanglement in non-local games and the hyperlinear profile of groups

William Slofstra[1] and Thomas Vidick[2]

[1]IQC, University of Waterloo

[2]Caltech

QIP, Delft, January 19th 2018

Fact: Entangled states generate non-local correlations

Fact: Entangled states generate non-local correlations

Questions:

Fact: Entangled states generate non-local correlations

Questions:

- *Direct problem*: Which correlations are achievable?

Fact: Entangled states generate non-local correlations

Questions:

- *Direct problem*: Which correlations are achievable?
- *Inverse problem*: Given a correlation, what is the class of states and measurements that realize it?

Fact: Entangled states generate non-local correlations

Questions:

- *Direct problem*: Which correlations are achievable?
- *Inverse problem*: Given a correlation, what is the class of states and measurements that realize it?
- How complex are they? Can we always find a realization in finite dimension? An *approximate* realization in finite dimension?

Fact: Entangled states generate non-local correlations

Questions:

- *Direct problem*: Which correlations are achievable?
- *Inverse problem*: Given a correlation, what is the class of states and measurements that realize it?
- How complex are they? Can we always find a realization in finite dimension? An *approximate* realization in finite dimension?
- What is a correlation anyways?

# Bipartite correlations

Two sites, $A$ and $B$

$n$ measurements at each site

each measurement has $m$ possible outcomes

## Bipartite correlations

Two sites, $A$ and $B$

$n$ measurements at each site

each measurement has $m$ possible outcomes

$p(a, b|x, y) =$ probability of outcome $(a, b)$
on measurements $(x, y)$

Correlation: set $\{p(a, b|x, y)\}$ of $n^2$ joint probability distributions

# Bipartite correlations

Two sites, $A$ and $B$

$n$ measurements at each site

each measurement has $m$ possible outcomes

$$p(a, b|x, y) = \text{probability of outcome } (a, b) \text{ on measurements } (x, y)$$

Correlation: set $\{p(a, b|x, y)\}$ of $n^2$ joint probability distributions

Example: $\left(\begin{array}{cc|cc} .5 & 0 & .25 & .25 \\ 0 & .5 & .25 & .25 \\ \hline 0 & .5 & .15 & .85 \\ .5 & 0 & .85 & .15 \end{array}\right)$ is a correlation with $n = m = 2$.

# Quantum correlations

$p(a, b|x, y) =$ probability of outcome $(a, b)$
on measurements $(x, y)$

Correlation: set $\{p(a, b|x, y)\}$ of $n^2$ joint probability distributions

# Quantum correlations

$p(a, b|x, y) =$ probability of outcome $(a, b)$
on measurements $(x, y)$

Correlation: set $\{p(a, b|x, y)\}$ of $n^2$ joint probability distributions

---

A correlation is *quantum* if it is of the form

$$p(a, b|x, y) = \langle\psi| M_x^a \otimes N_y^b |\psi\rangle$$

for some

- $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and
- measurements $\{M_x^a\}$ on $\mathcal{H}_A$, $\{N_y^b\}$ on $\mathcal{H}_B$

Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ can be finite or infinite-dimensional

## Correlation sets

A correlation is *quantum* if it is of the form

$$p(a, b|x, y) = \langle\psi| M_x^a \otimes N_y^b |\psi\rangle$$

for some $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and meas. $\{M_x^a\}$ on $\mathcal{H}_A$, $\{N_y^b\}$ on $\mathcal{H}_B$

---

## Correlation sets

A correlation is *quantum* if it is of the form

$$p(a, b|x, y) = \langle\psi| M_x^a \otimes N_y^b |\psi\rangle$$

for some $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and meas. $\{M_x^a\}$ on $\mathcal{H}_A$, $\{N_y^b\}$ on $\mathcal{H}_B$

---

$C_q(n, m) =$ set of all quantum correlations ($n$ measurements, $m$ outcomes) where $\mathcal{H}_A$, $\mathcal{H}_B$ are finite

$C_{qs}(n, m) =$ set of all quantum correlations ($n$ measurements, $m$ outcomes) where $\mathcal{H}_A$, $\mathcal{H}_B$ are possibly $\infty$-dimensional

Both sets $C_q(n, m)$ and $C_{qs}(n, m)$ are convex

## Correlation sets

Both sets $C_q(n, m)$ and $C_{qs}(n, m)$ are convex

This requires to keep $\dim \mathcal{H}_A$ and $\dim \mathcal{H}_B$ unbounded!

---

## Correlation sets

Both sets $C_q(n,m)$ and $C_{qs}(n,m)$ are <u>convex</u>

This requires to keep $\dim \mathcal{H}_A$ and $\dim \mathcal{H}_B$ unbounded!

---

Example: the quantum set $C_q(2,2)$ contains

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} .5 & 0 & 0 & .5 \\ 0 & .5 & .5 & 0 \\ \hline 0 & .5 & .5 & 0 \\ .5 & 0 & 0 & .5 \end{array}\right),$$

but the third correlation requires $\dim(\mathcal{H}) \geq 2$ (one bit of shared randomness is enough).

# Correlation sets

Both sets $C_q(n,m)$ and $C_{qs}(n,m)$ are convex

This requires to keep $\dim \mathcal{H}_A$ and $\dim \mathcal{H}_B$ unbounded!

---

Example: the quantum set $C_q(2,2)$ contains

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} .5 & 0 & 0 & .5 \\ 0 & .5 & .5 & 0 \\ \hline 0 & .5 & .5 & 0 \\ .5 & 0 & 0 & .5 \end{array}\right),$$

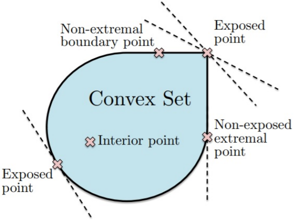
but the third correlation requires $\dim(\mathcal{H}) \geq 2$ (one bit of shared randomness is enough).

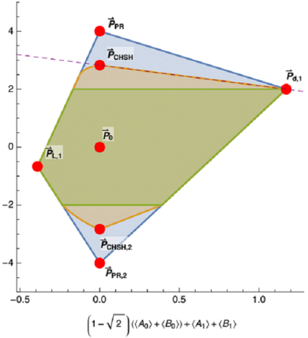
$C_q$ and $C_{qs}$ have the same closure.

# The quantum set

Research program: characterize the set $C_q(n, m)$ for each $n, m$

[Goh et. al. '17] $n = m = 2$ case has complex geometry! Flat boundaries, curved boundaries, non-exposed extreme points, etc.

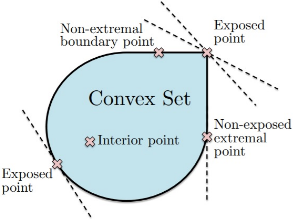


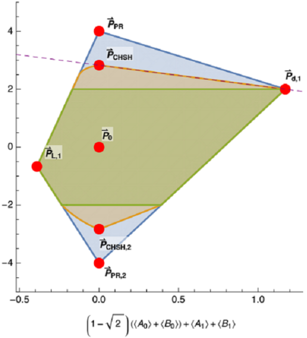



# The quantum set

Research program: characterize the set $C_q(n,m)$ for each $n,m$

[Goh et. al. '17] $n=m=2$ case has complex geometry! Flat boundaries, curved boundaries, non-exposed extreme points, etc.




*But*: $C_q(2,2)$ is closed, and in fact $\dim(\mathcal{H})=2$ is always sufficient.

## The quantum set

Research program: characterize the set $C_q(n, m)$ for each $n, m$

*Is $C_q(n, m)$ closed for general $n, m$?*

## The quantum set

Research program: characterize the set $C_q(n, m)$ for each $n, m$

*Is $C_q(n, m)$ closed for general $n, m$?*

Question goes back to [Tsirelson '93], who initiated the systematic study of $C_q$, $C_{qs}$ and $C_{qc}$.

## The quantum set

Research program: characterize the set $C_q(n, m)$ for each $n, m$

$$\textit{Is } C_q(n, m) \textit{ closed for general } n, m?$$

Question goes back to [Tsirelson '93], who initiated the systematic study of $C_q$, $C_{qs}$ and $C_{qc}$.

Trivial inclusions: $C_c \subsetneq C_q \subseteq C_{qs} \subseteq C_{qc}$.

## The quantum set

Research program: characterize the set $C_q(n, m)$ for each $n, m$

*Is $C_q(n, m)$ closed for general $n, m$?*

Question goes back to [Tsirelson '93], who initiated the systematic study of $C_q$, $C_{qs}$ and $C_{qc}$.

Trivial inclusions: $C_c \subsetneq C_q \subseteq C_{qs} \subseteq C_{qc}$.

Tsirelson assumed $C_{qs} = C_{qc}$ and asked if $C_q = C_{qs}$.

## The quantum set

Research program: characterize the set $C_q(n,m)$ for each $n, m$

*Is $C_q(n,m)$ closed for general $n, m$?*

Question goes back to [Tsirelson '93], who initiated the systematic study of $C_q$, $C_{qs}$ and $C_{qc}$.

Trivial inclusions: $C_c \subsetneq C_q \subseteq C_{qs} \subseteq C_{qc}$.

Tsirelson assumed $C_{qs} = C_{qc}$ and asked if $C_q = C_{qs}$.

[Slofstra '16] $C_q(n,m) \subsetneq C_{qc}(n,m)$ (known: $C_{qc}$ is closed)

## The quantum set

Research program: characterize the set $C_q(n, m)$ for each $n, m$

*Is $C_q(n, m)$ closed for general $n, m$?*

Question goes back to [Tsirelson '93], who initiated the systematic study of $C_q$, $C_{qs}$ and $C_{qc}$.

Trivial inclusions: $C_c \subsetneq C_q \subseteq C_{qs} \subseteq C_{qc}$.

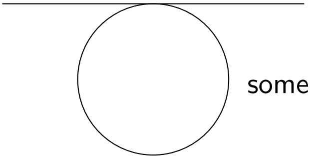
Tsirelson assumed $C_{qs} = C_{qc}$ and asked if $C_q = C_{qs}$.

[Slofstra '16] $C_q(n, m) \subsetneq C_{qc}(n, m)$ (known: $C_{qc}$ is closed)

*Is $C_q$ closed?*

# Non-local games (aka Bell-type experiments)

Closures of convex sets can be described by separating hyperplanes



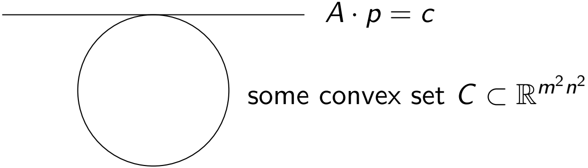
$A \cdot p = c$

some convex set $C \subset \mathbb{R}^{m^2 n^2}$

$$\text{where } A \cdot p := \sum_{a,b,x,y} A(a,b,x,y)\, p(a,b|x,y)$$

# Non-local games (aka Bell-type experiments)

Closures of convex sets can be described by separating hyperplanes




$$\text{where } A \cdot p := \sum_{a,b,x,y} A(a,b,x,y)p(a,b|x,y)$$

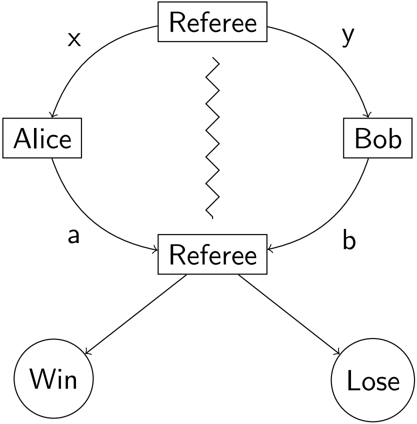
To describe $C$, need to be able to find $c = \sup\{A \cdot p : p \in C\}$

*Quantum value of* $A$: $\omega_q(A) = \sup\{A \cdot p : p \in C_q(n,m)\}$

($\omega_q(A)$ is the maximal quantum violation of a Bell inequality)

# Non-local games (aka Bell-type experiments)

For $C_q(m, n)$, helpful to think of supporting hyperplanes as games



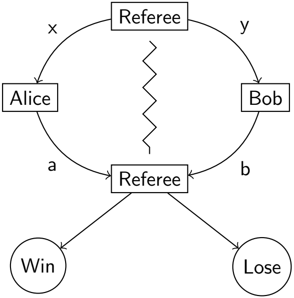

Win/lose based on outputs $a, b$ and inputs $x, y$

Alice and Bob must cooperate to win

Winning conditions known in advance

Players cannot communicate while the game is in progress

# Non-local games ct'd




Non-local game given by:

Probability distribution $\pi(x, y)$ on questions

Pay-off function:
$V(a, b|x, y) = 1$ if answers $(a, b)$ win on questions $(x, y)$,
$V(a, b|x, y) = 0$ otherwise

Quantum value: $\omega_q(G) =$ optimal winning probability when players can share an entangled state

$$\omega_q(G) = \sup \left\{ \sum_{a,b,x,y} \pi(x, y) V(a, b|x, y) p(a, b|x, y) : p \in C_q(n, m) \right\}$$

## Explicit lower bounds on entanglement

*Can we find a game $G$ and an $\epsilon \geq 0$ such that playing $G$ with success probability $\geq \omega_q(G) - \epsilon$ requires Schmidt rank $\geq d$?*

If $d(\epsilon, n, m) \to \infty$ for fixed $n, m$ as $\epsilon \to 0$ then $C_q$ is not closed.

# Explicit lower bounds on entanglement

*Can we find a game $G$ and an $\epsilon \geq 0$ such that playing $G$ with success probability $\geq \omega_q(G) - \epsilon$ requires Schmidt rank $\geq d$?*

If $d(\epsilon, n, m) \to \infty$ for fixed $n, m$ as $\epsilon \to 0$ then $C_q$ is not closed.

- [Brunner et. al '08] Asked original question

# Explicit lower bounds on entanglement

*Can we find a game $G$ and an $\epsilon \geq 0$ such that playing $G$ with success probability $\geq \omega_q(G) - \epsilon$ requires Schmidt rank $\geq d$?*

If $d(\epsilon, n, m) \to \infty$ for fixed $n, m$ as $\epsilon \to 0$ then $C_q$ is not closed.

- [Brunner et. al '08] Asked original question
- [Junge-Palazuelos '11] $m = n \approx \sqrt{d}$, multiplicative $O(d)$

# Explicit lower bounds on entanglement

*Can we find a game $G$ and an $\epsilon \geq 0$ such that playing $G$ with success probability $\geq \omega_q(G) - \epsilon$ requires Schmidt rank $\geq d$?*

If $d(\epsilon, n, m) \to \infty$ for fixed $n, m$ as $\epsilon \to 0$ then $C_q$ is not closed.

- [Brunner et. al '08] Asked original question
- [Junge-Palazuelos '11] $m = n \approx \sqrt{d}$, multiplicative $O(d)$
- [Ostrev-V., Chao et al. '16] $m = 2$, $n \approx (\log d)^2$, any $\epsilon = O(n^{-5/2})$.

# Explicit lower bounds on entanglement

*Can we find a game $G$ and an $\epsilon \geq 0$ such that playing $G$ with success probability $\geq \omega_q(G) - \epsilon$ requires Schmidt rank $\geq d$?*

If $d(\epsilon, n, m) \to \infty$ for fixed $n, m$ as $\epsilon \to 0$ then $C_q$ is not closed.

- [Brunner et. al '08] Asked original question
- [Junge-Palazuelos '11] $m = n \approx \sqrt{d}$, multiplicative $O(d)$
- [Ostrev-V., Chao et al. '16] $m = 2$, $n \approx (\log d)^2$, any $\epsilon = O(n^{-5/2})$.
- [Coladangelo-Stark '17] $m \approx d^2$, $n =$constant, any $\epsilon = O(d^{-3})$.

# Explicit lower bounds on entanglement

*Can we find a game $G$ and an $\epsilon \geq 0$ such that playing $G$ with success probability $\geq \omega_q(G) - \epsilon$ requires Schmidt rank $\geq d$?*

If $d(\epsilon, n, m) \to \infty$ for fixed $n, m$ as $\epsilon \to 0$ then $C_q$ is not closed.

- [Brunner et. al '08] Asked original question
- [Junge-Palazuelos '11] $m = n \approx \sqrt{d}$, multiplicative $O(d)$
- [Ostrev-V., Chao et al. '16] $m = 2$, $n \approx (\log d)^2$, any $\epsilon = O(n^{-5/2})$.
- [Coladangelo-Stark '17] $m \approx d^2$, $n =$constant, any $\epsilon = O(d^{-3})$.
- [Natarajan-V. '18] $m =$constant, $n = (\log d)^c$, any small enough constant $\epsilon$.

# Explicit lower bounds on entanglement

*Can we find a game $G$ and an $\epsilon \geq 0$ such that playing $G$ with success probability $\geq \omega_q(G) - \epsilon$ requires Schmidt rank $\geq d$?*

If $d(\epsilon, n, m) \to \infty$ for fixed $n, m$ as $\epsilon \to 0$ then $C_q$ is not closed.

- [Pál-Vértesi '10] Bell inequality $I_{3322}$ with $n = 3, m = 2$. Conjecture maximal violation requires infinite-dimensional entanglement.

# Explicit lower bounds on entanglement

*Can we find a game $G$ and an $\epsilon \geq 0$ such that playing $G$ with success probability $\geq \omega_q(G) - \epsilon$ requires Schmidt rank $\geq d$?*

If $d(\epsilon, n, m) \to \infty$ for fixed $n, m$ as $\epsilon \to 0$ then $C_q$ is not closed.

- [Pál-Vértesi '10] Bell inequality $I_{3322}$ with $n = 3, m = 2$. Conjecture maximal violation requires infinite-dimensional entanglement.
- [Leung-Toner-Watrous '08, Mančinska-V. '14, Regev-V. '15, Coladangelo-Stark'17] propose variants of non-local games: quantum questions/answers, or infinite question/answer sets. Typical scaling: $d(\epsilon) \asymp 2^{\epsilon^{-c}}$.

# Explicit *upper bounds* on entanglement

*Given a game $G$ and $\epsilon \geq 0$, can we find a $d_{min}$ such that $G$ can be played with success $\geq \omega_q(G) - \epsilon$ using Schmidt rank $\leq d_{min}$?*

## Explicit *upper bounds* on entanglement

*Given a game $G$ and $\epsilon \geq 0$, can we find a $d_{min}$ such that $G$ can be played with success $\geq \omega_q(G) - \epsilon$ using Schmidt rank $\leq d_{min}$?*

?

# Explicit *upper bounds* on entanglement

*Given a game $G$ and $\epsilon \geq 0$, can we find a $d_{min}$ such that $G$ can be played with success $\geq \omega_q(G) - \epsilon$ using Schmidt rank $\leq d_{min}$?*

?

By a compactness argument, such a $d_{min} = d_{min}(n, m, \epsilon)$ exists for any correlation in $C_q(n, m)$.

# Explicit *upper bounds* on entanglement

*Given a game $G$ and $\epsilon \geq 0$, can we find a $d_{min}$ such that $G$ can be played with success $\geq \omega_q(G) - \epsilon$ using Schmidt rank $\leq d_{min}$?*

?

By a compactness argument, such a $d_{min} = d_{min}(n, m, \epsilon)$ exists for any correlation in $C_q(n, m)$.

If Connes' Embedding Conjecture holds, then $d_{min}$ is computable.

## Results

**Theorem (Slofstra, arXiv:1703.08618)**

*There is a finite game $G$ such that $\omega_q(G) = 1$, but which cannot be played perfectly using any correlation in $C_q$ (or even $C_{qs}$).*

In other words, if $E(G, \epsilon)$ is the Schmidt rank required to achieve success probability $\omega_q(G) - \epsilon$, then $E(G, \epsilon) \to +\infty$ as $\epsilon \to 0$

## Results

**Theorem (Slofstra, arXiv:1703.08618)**

*There is a finite game $G$ such that $\omega_q(G) = 1$, but which cannot be played perfectly using any correlation in $C_q$ (or even $C_{qs}$).*

In other words, if $E(G, \epsilon)$ is the Schmidt rank required to achieve success probability $\omega_q(G) - \epsilon$, then $E(G, \epsilon) \to +\infty$ as $\epsilon \to 0$

**Corollary (Slofstra)**

*$C_q(n, m)$ and $C_{qs}(n, m)$ are not closed for some finite $n, m$.*

Proof yields $n \approx 240$, $m = 8$.

[Dykema-Paulsen-Prakash '17] $n = 5$, $m = 2$

# Results

**Theorem (Slofstra, arXiv:1703.08618)**

*There is a finite game $G$ such that $\omega_q(G) = 1$, but which cannot be played perfectly using any correlation in $C_q$ (or even $C_{qs}$).*

$E(G, \epsilon)$: Schmidt rank required to achieve success $\omega_q(G) - \epsilon$

**Theorem (Slofstra-V., arXiv:1711.10676)**

*There is a finite game $G$ and constants $C, C', k > 0$ such that*

$$\forall \epsilon \geq 0 \qquad \frac{C}{\epsilon^{1/k}} \leq E(G, \epsilon) \leq \frac{C'}{\epsilon^{1/2}} .$$

# From groups to games

Finitely presented group: $K = \langle S;\, R\rangle$.

$S$ : finite set of generators

$R$ : finite set of relations

## From groups to games

Finitely presented group: $K = \langle S;\ R \rangle$.

$S$ : finite set of generators

$R$ : finite set of relations

*Example*: Weyl-Heisenberg group

$$K = \langle J, X, Z;\ J^2 = X^2 = Z^2 = 1,\\ [J, X] = [J, Z] = 1,\ J[X, Z] = 1 \rangle.$$

## From groups to games

Finitely presented group: $K = \langle S;\ R \rangle$.

$S$ : finite set of generators

$R$ : finite set of relations

*Example*: Weyl-Heisenberg group

$$K = \langle J, X, Z;\ J^2 = X^2 = Z^2 = 1,\\ [J, X] = [J, Z] = 1,\ J[X, Z] = 1 \rangle.$$

Group representation:

Map $\phi : S \to U(\mathbb{C}^d)$ such that $\phi(r) = I$ for all $r \in R$.

# From groups to games

Finitely presented group: $K = \langle S;\ R \rangle$.

$S$ : finite set of generators

$R$ : finite set of relations

*Example*: Weyl-Heisenberg group

$$K = \langle J, X, Z;\ J^2 = X^2 = Z^2 = 1, \\ [J, X] = [J, Z] = 1,\ J[X, Z] = 1 \rangle.$$

Group representation:

Map $\phi : S \to U(\mathbb{C}^d)$ such that $\phi(r) = I$ for all $r \in R$.

*Example*: $\phi(J) = -I$, $\phi(X) = \sigma_X$, $\phi(Z) = \sigma_Z$ (non-trivial)

# From groups to games

Finitely presented group: $K = \langle S;\ R \rangle$.

$S$ : finite set of generators

$R$ : finite set of relations

*Example*: Weyl-Heisenberg group

$$K = \langle J, X, Z;\ J^2 = X^2 = Z^2 = 1, \\ [J,X] = [J,Z] = 1,\ J[X,Z] = 1 \rangle.$$

Group representation:

Map $\phi : S \to U(\mathbb{C}^d)$ such that $\phi(r) = I$ for all $r \in R$.

*Example*: $\phi(J) = -I$, $\phi(X) = \sigma_X$, $\phi(Z) = \sigma_Z$ (non-trivial)

*Example*: $\phi(J) = I$, $\phi(X) = I$, $\phi(Z) = I$ (trivial)

# From groups to games

Finitely presented group: $K = \langle S;\, R \rangle$.
$S$: generators. $R$: relations.

Group representation:

Map $\phi : S \to U(\mathbb{C}^d)$ such that $\phi(r) = I$ for all $r \in R$.

---

# From groups to games

Finitely presented group: $K = \langle S;\, R \rangle$.
$S$: generators. $R$: relations.

Group representation:
Map $\phi : S \to U(\mathbb{C}^d)$ such that $\phi(r) = I$ for all $r \in R$.

---

$$\begin{array}{c} \text{Finitely-presented group} \\ K = \langle S;\, R \rangle \end{array} \longleftrightarrow \text{Finite game } G$$

# From groups to games

Finitely presented group: $K = \langle S;\, R\rangle$.
$S$: generators. $R$: relations.

Group representation:
Map $\phi : S \to U(\mathbb{C}^d)$ such that $\phi(r) = I$ for all $r \in R$.

---

$$\begin{array}{ccc}
\text{Finitely-presented group } K = \langle S;\, R\rangle & \longleftrightarrow & \text{Finite game } G \\
\text{Representation (of dimension } d\text{)} & \longleftrightarrow & \text{Winning strategy (of dimension } d\text{)}
\end{array}$$

# From groups to games

Finitely presented group: $K = \langle S;\, R\rangle$.
$S$: generators. $R$: relations.

Group representation:
Map $\phi : S \to U(\mathbb{C}^d)$ such that $\phi(r) = I$ for all $r \in R$.

---

| | | |
|---|---|---|
| Finitely-presented group $K = \langle S;\, R\rangle$ | $\longleftrightarrow$ | Finite game $G$ |
| Representation (of dimension $d$) | $\longleftrightarrow$ | Winning strategy (of dimension $d$) |
| Fancy group | $\longleftrightarrow$ | Interesting game |

# From groups to games

Finitely presented group: $K = \langle S;\, R \rangle$.
$S$: generators. $R$: relations.

Group representation:
Map $\phi : S \to U(\mathbb{C}^d)$ such that $\phi(r) = I$ for all $r \in R$.

---

| | | |
|---|---|---|
| Finitely-presented group $K = \langle S;\, R \rangle$ | $\longleftrightarrow$ | Finite game $G$ |
| Representation (of dimension $d$) | $\longleftrightarrow$ | Winning strategy (of dimension $d$) |
| Fancy group | $\longleftrightarrow$ | Interesting game |

*Caveat:* Only *non-trivial* reps should give winning strategies!

## From groups to games

Finitely presented group: $K = \langle S;\, R \rangle$.
$S$: generators. $R$: relations.

Group representation:
Map $\phi : S \to U(\mathbb{C}^d)$ such that $\phi(r) = I$ for all $r \in R$.

---

Finitely-presented group
$K = \langle S;\, R \rangle$ $\longleftrightarrow$ Finite game $G$
$J$ a central involution

# From groups to games

Finitely presented group: $K = \langle S;\, R\rangle$.
$S$: generators. $R$: relations.

Group representation:
Map $\phi : S \to U(\mathbb{C}^d)$ such that $\phi(r) = I$ for all $r \in R$.

---

| | | |
|---|---|---|
| Finitely-presented group<br>$K = \langle S;\, R\rangle$<br>$J$ a central involution | $\longleftrightarrow$ | Finite game $G$ |
| Representation of<br>dimension $d$<br>such that $\phi(J) = -I$ | $\longleftrightarrow$ | Winning strategy of<br>dimension $d$ |

# From groups to games

Finitely presented group: $K = \langle S;\, R \rangle$.
$S$: generators. $R$: relations.

Group representation:
Map $\phi : S \to U(\mathbb{C}^d)$ such that $\phi(r) = I$ for all $r \in R$.

---

| | | |
|---|---|---|
| Finitely-presented group<br>$K = \langle S;\, R \rangle$<br>$J$ a central involution | $\longleftrightarrow$ | Finite game $G$ |
| Representation of dimension $d$ such that $\phi(J) = -I$ | $\longleftrightarrow$ | Winning strategy of dimension $d$ |
| $\epsilon$-approximate rep. such that $\phi(J) \approx_\epsilon -I$ | $\longleftrightarrow$ | Strategy with success $(1-\epsilon)$ |

# From groups to games

$K = \langle S; R \rangle$
$J \in K$ central involution $\longleftrightarrow$ Game $G$

Representation $\phi$
such that $\phi(J) = -I$ $\longleftrightarrow$ Winning strategy in $G$

---

# From groups to games

$K = \langle S;\ R \rangle$
$J \in K$ central involution $\longleftrightarrow$ Game $G$

Representation $\phi$
such that $\phi(J) = -I$ $\longleftrightarrow$ Winning strategy in $G$

---

Dream application:

Weyl-Heisenberg group $\longleftrightarrow$ Magic Square game

$$K = \langle J, X, Z;\ J^2 = X^2 = Z^2 = 1,\ [J, X] = [J, Z] = 1,\ [X, Z] = J \rangle \longleftrightarrow \begin{bmatrix} XI & IX & XX \\ IZ & ZI & ZZ \\ XZ & ZX & YY \end{bmatrix}$$

Pauli representation
$\sigma_X, \sigma_Z$ $\longleftrightarrow$ Perfect 2-qubit strategy

# From groups to games

Linear System Games (Cleve-Mittal '15):

"Solution group" $\Gamma$, $J' \in \Gamma$ $\longleftrightarrow$ Game $G$

$\phi'$ such that $\phi'(J') = -I$ $\longleftrightarrow$ Winning strategy in $G$

# From groups to games

Linear System Games (Cleve-Mittal '15):

"Solution group" $\Gamma$, $J' \in \Gamma$ $\longleftrightarrow$ Game $G$

$\phi'$ such that $\phi'(J') = -I$ $\longleftrightarrow$ Winning strategy in $G$

- $J'$ non-trivial in $\Gamma$ $\leftrightarrow$ perfect commuting strategy ($C_{qc}$)
- $J'$ non-trivial in finite-dim. rep. $\leftrightarrow$ perfect finite strategy ($C_q$)
  $\leftrightarrow$ perfect infinite strategy ($C_{qs}$)

## From groups to games

Linear System Games (Cleve-Mittal '15):

"Solution group" $\Gamma$, $J' \in \Gamma$ $\longleftrightarrow$ Game $G$

$\phi'$ such that $\phi'(J') = -I$ $\longleftrightarrow$ Winning strategy in $G$

- $J'$ non-trivial in $\Gamma \leftrightarrow$ perfect commuting strategy ($C_{qc}$)
- $J'$ non-trivial in finite-dim. rep. $\leftrightarrow$ perfect finite strategy ($C_q$)
  $\leftrightarrow$ perfect infinite strategy ($C_{qs}$)

Program to separate $C_{qs}$ from $C_{qc}$: find

- A solution group $\Gamma$
- $J' \in \Gamma$ a non-trivial group element
- $J'$ is trivial in all finite-dim representations

# From groups to games

Universal embedding theorem (Slofstra '16):

$$\begin{array}{ccc} K = \langle S;\, R\rangle & & \text{Solution group } \Gamma \\ J \in K \text{ central involution} & \longleftrightarrow & J' \in \Gamma \\ \text{Representation } \phi & & \text{Representation } \phi' \\ \text{such that } \phi(J) = -I & \longleftrightarrow & \text{such that } \phi'(J') = -I \end{array}$$

# From groups to games

Universal embedding theorem (Slofstra '16):

| | | |
|---|---|---|
| $K = \langle S;\, R \rangle$<br>$J \in K$ central involution | $\longleftrightarrow$ | Solution group $\Gamma$<br>$J' \in \Gamma$ |
| Representation $\phi$<br>such that $\phi(J) = -I$ | $\longleftrightarrow$ | Representation $\phi'$<br>such that $\phi'(J') = -I$ |

Upside:

- Gives canonical structure of group presentation
- Preserves non-triviality of representations

# From groups to games

Universal embedding theorem (Slofstra '16):

| | | |
|---|---|---|
| $K = \langle S;\ R \rangle$<br>$J \in K$ central involution | $\longleftrightarrow$ | Solution group $\Gamma$<br>$J' \in \Gamma$ |
| Representation $\phi$<br>such that $\phi(J) = -I$ | $\longleftrightarrow$ | Representation $\phi'$<br>such that $\phi'(J') = -I$ |

Upside:

- Gives canonical structure of group presentation
- Preserves non-triviality of representations

Downside:

- No control of dimension of $\phi'$
- No control of approximate representations

# From groups to games

Universal embedding theorem (Slofstra '16):

| | | |
|---|---|---|
| $K = \langle S;\, R \rangle$<br>$J \in K$ central involution | $\longleftrightarrow$ | Solution group $\Gamma$<br>$J' \in \Gamma$ |
| Representation $\phi$<br>such that $\phi(J) = -I$ | $\longleftrightarrow$ | Representation $\phi'$<br>such that $\phi'(J') = -I$ |

## From groups to games

Universal embedding theorem (Slofstra '16):

| | | |
|---|---|---|
| $K = \langle S;\ R\rangle$<br>$J \in K$ central involution | $\longleftrightarrow$ | Solution group $\Gamma$<br>$J' \in \Gamma$ |
| Representation $\phi$<br>such that $\phi(J) = -I$ | $\longleftrightarrow$ | Representation $\phi'$<br>such that $\phi'(J') = -I$ |

Linear System Games (Cleve-Mittal '15):

Solution group $\Gamma$, $J' \in \Gamma$ $\longleftrightarrow$ Game $G$

$\phi'$ such that $\phi'(J') = -I$ $\longleftrightarrow$ Winning strategy in $G$

- $J$ non-trivial in $\Gamma$ $\leftrightarrow$ perfect commuting strategy ($C_{qc}$)
- $J$ trivial in finite-dim. rep. $\leftrightarrow$ no perfect finite strategy ($C_q$)
  $\leftrightarrow$ no perfect infinite strategy ($C_{qs}$)

# An interesting group

$$K = \langle a, b, c, x, y : \ xyx^{-1} = y^2, \ xcx^{-1} = c,$$
$$yay^{-1} = b, \ yby^{-1} = a, \ c = ab, \ a^2 = b^2 = c^2 = e \rangle \ .$$

# An interesting group

$$K = \langle a, b, c, x, y : \; xyx^{-1} = y^2, \; xcx^{-1} = c, \\ yay^{-1} = b, \; yby^{-1} = a, \; c = ab, \; a^2 = b^2 = c^2 = e \rangle \; .$$

$K$ has the following properties:

(0) The element $c$ is non-trivial in $K$.

(i) Any finite-dimensional representation $\phi$ sends $c$ to $I$.

# An interesting group

$$K = \langle a, b, c, x, y : \; xyx^{-1} = y^2, \; xcx^{-1} = c,$$
$$yay^{-1} = b, \; yby^{-1} = a, \; c = ab, \; a^2 = b^2 = c^2 = e \rangle \; .$$

$K$ has the following properties:

(0) The element $c$ is non-trivial in $K$.

(i) Any finite-dimensional representation $\phi$ sends $c$ to $I$.

Finitely presented group $K$, element $J = c$
+ Slofstra's embedding theorem
+ Cleve-Mittal mapping from solution group to games:

$$C_{qs} \subsetneq C_{qc} \; .$$

# An interesting group

$$K = \langle a, b, c, x, y : \ xyx^{-1} = y^2, \ xcx^{-1} = c,$$
$$yay^{-1} = b, \ yby^{-1} = a, \ c = ab, \ a^2 = b^2 = c^2 = e\rangle \ .$$

$K$ has the following properties:

(0) The element $c$ is non-trivial in $K$.

(i) Any finite-dimensional representation $\phi$ sends $c$ to $I$.

# An interesting group

$$K = \langle a, b, c, x, y : \ xyx^{-1} = y^2, \ xcx^{-1} = c,$$
$$yay^{-1} = b, \ yby^{-1} = a, \ c = ab, \ a^2 = b^2 = c^2 = e \rangle \ .$$

$K$ has the following properties:

(0) ~~The element $c$ is non-trivial in $K$.~~

(i) Any finite-dimensional representation $\phi$ sends $c$ to $I$.

(ii) For any $\epsilon > 0$ there is a $d$ and $\phi : K \to U(\mathbb{C}^d)$ such that $\|\phi(c) - I\|_f > 2 - \epsilon$ and $\|\phi(r) - I\|_f < \epsilon$ for all $r \in R$.

# An interesting group

$$K = \langle a, b, c, x, y : \; xyx^{-1} = y^2, \; xcx^{-1} = c, \\ yay^{-1} = b, \; yby^{-1} = a, \; c = ab, \; a^2 = b^2 = c^2 = e \rangle \; .$$

$K$ has the following properties:

(0) ~~The element $c$ is non-trivial in $K$.~~

(i) Any finite-dimensional representation $\phi$ sends $c$ to $I$.

(ii) For any $\epsilon > 0$ there is a $d$ and $\phi : K \to U(\mathbb{C}^d)$ such that $\|\phi(c) - I\|_f > 2 - \epsilon$ and $\|\phi(r) - I\|_f < \epsilon$ for all $r \in R$.

Wanted: embedding theorem (group $K \mapsto$ game $G$) s.t.

- Using (i), $G$ has no perfect finite-dimensional strategy;
- Using (ii), $\forall \epsilon > 0$, $G$ has an $\epsilon$-optimal strategy in finite dim $d$.

## Quantitative embeddings

**Theorem (Slofstra)**

*$K$ embeds in a solution group $\Gamma$, in such a way that approximate representations of $K$ lift to approximate representations of $\Gamma$.*

Theorem is more general: applies to "linear-plus-conjugacy" games.

# Quantitative embeddings

**Theorem (Slofstra)**

*$K$ embeds in a solution group $\Gamma$, in such a way that approximate representations of $K$ lift to approximate representations of $\Gamma$.*

Theorem is more general: applies to "linear-plus-conjugacy" games.

Corollary: Finite game $G$ such that $G$ has no perfect finite-dimensional strategy, but $\omega_q(G) = 1$.

Corollary: The sets $C_q$ and $C_{qs}$ are not closed.

# Quantitative embeddings

Hyperlinear profile $\mathrm{hlp}(w, \epsilon)$: smallest dimension $d$ such that there is a $d$-dimensional $\epsilon$-representation $\phi$ with $\|\phi(w) - 1\|_f \geq 2 - \epsilon$.

## Quantitative embeddings

Hyperlinear profile $\mathrm{hlp}(w,\epsilon)$: smallest dimension $d$ such that there is a $d$-dimensional $\epsilon$-representation $\phi$ with $\|\phi(w)-1\|_f \geq 2-\epsilon$.

**Theorem (Slofstra-V.)**

*Let $\Gamma$ be a solution group and $G$ the associated game.*

| | | |
|---|---|---|
| *$d$-dimensional $\epsilon$-representations of $\Gamma$ with $J \mapsto -1$* | $\longleftrightarrow$ | *$O(\epsilon^2)$-perfect strategies for $G$ using a maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^d$* |

# Quantitative embeddings

Hyperlinear profile $\mathrm{hlp}(w, \epsilon)$: smallest dimension $d$ such that there is a $d$-dimensional $\epsilon$-representation $\phi$ with $\|\phi(w) - 1\|_f \geq 2 - \epsilon$.

**Theorem (Slofstra-V.)**

*Let $\Gamma$ be a solution group and $G$ the associated game.*

*$d$-dimensional $\epsilon$-representations of $\Gamma$ with $J \mapsto -1$* $\longleftrightarrow$ *$O(\epsilon^2)$-perfect strategies for $G$ using a maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^d$*

Corollary: Achieving success probability $1 - \epsilon$ in $G$ with a maximally entangled state requires local Hilbert space dimension $\geq \mathrm{hlp}(J, \Theta(\sqrt{\epsilon}))$.

# An interesting group

$$K = \langle a, b, c, x, y : \; xyx^{-1} = y^2, \; xcx^{-1} = c,$$
$$yay^{-1} = b, \; yby^{-1} = a, \; c = ab, \; a^2 = b^2 = c^2 = e \rangle \; .$$

$\mathrm{hlp}(w, \epsilon)$: smallest dimension $d$ such that there is a $d$-dimensional $\epsilon$-representation $\phi$ with $\|\phi(w) - 1\|_f \geq 2 - \epsilon$.

**Theorem (Slofstra-V.)**

*The group $K$ has hyperlinear profile* $\mathrm{hlp}(c, \epsilon) = \Theta(1/\epsilon^{2/3})$.

# An interesting group

$$K = \langle a, b, c, x, y : \; xyx^{-1} = y^2, \; xcx^{-1} = c,$$
$$yay^{-1} = b, \; yby^{-1} = a, \; c = ab, \; a^2 = b^2 = c^2 = e \rangle \; .$$

$\mathrm{hlp}(w, \epsilon)$: smallest dimension $d$ such that there is a $d$-dimensional $\epsilon$-representation $\phi$ with $\|\phi(w) - 1\|_f \geq 2 - \epsilon$.

**Theorem (Slofstra-V.)**

*The group $K$ has hyperlinear profile* $\mathrm{hlp}(c, \epsilon) = \Theta(1/\epsilon^{2/3})$.

Corollary: There is a game $G$ such that success probability $1 - \epsilon$ requires entanglement of dimension $\geq \frac{C}{\epsilon^{1/k}}$, $2 \leq k \leq 20$.

(We also show it is possible to succeed with probability $\geq 1 - \epsilon$ using a maximally entangled state of dimension $\leq \frac{C'}{\epsilon^{1/2}}$.)

## An interesting group

$$K = \langle a, b, c, x, y : \; xyx^{-1} = y^2, \; xcx^{-1} = c, \; yay^{-1} = b, \; yby^{-1} = a, \; c = ab, \; a^2 = b^2 = c^2 = e \rangle \; .$$

**Theorem (Slofstra-V.)**

*The group $K$ has hyperlinear profile* $\mathrm{hlp}(c, \epsilon) = \Theta(1/\epsilon^{2/3})$.

# An interesting group

$$K = \langle a, b, c, x, y : \ xyx^{-1} = y^2, \ xcx^{-1} = c,$$
$$yay^{-1} = b, \ yby^{-1} = a, \ c = ab, \ a^2 = b^2 = c^2 = e \rangle \ .$$

**Theorem (Slofstra-V.)**

*The group $K$ has hyperlinear profile* $\mathrm{hlp}(c, \epsilon) = \Theta(1/\epsilon^{2/3})$.

Proof idea:

1. Find basis s.t. $\phi(y) = \begin{pmatrix} 0 & I \\ U & 0 \end{pmatrix}$, for some unitary $U$.

# An interesting group

$$K = \langle a, b, c, x, y : \; xyx^{-1} = y^2, \; xcx^{-1} = c,$$
$$yay^{-1} = b, \; yby^{-1} = a, \; c = ab, \; a^2 = b^2 = c^2 = e \rangle \; .$$

**Theorem (Slofstra-V.)**

*The group $K$ has hyperlinear profile* $\mathrm{hlp}(c, \epsilon) = \Theta(1/\epsilon^{2/3})$.

Proof idea:

1. Find basis s.t. $\phi(y) = \begin{pmatrix} 0 & I \\ U & 0 \end{pmatrix}$, for some unitary $U$.
2. Use $xyx^{-1} = y^2$ to argue:

$$e^{2i\pi\theta} \in \mathrm{Spec}(U) \implies e^{2i\pi\frac{\theta}{2}}, e^{2i\pi\left(\frac{\theta}{2}+\frac{1}{2}\right)} \in \mathrm{Spec}(U)$$

## An interesting group

$$K = \langle a, b, c, x, y : \; xyx^{-1} = y^2, \; xcx^{-1} = c,$$
$$yay^{-1} = b, \; yby^{-1} = a, \; c = ab, \; a^2 = b^2 = c^2 = e\rangle \; .$$

**Theorem (Slofstra-V.)**

*The group $K$ has hyperlinear profile* $\mathrm{hlp}(c, \epsilon) = \Theta(1/\epsilon^{2/3})$.

Proof idea:

1. Find basis s.t. $\phi(y) = \begin{pmatrix} 0 & I \\ U & 0 \end{pmatrix}$, for some unitary $U$.
2. Use $xyx^{-1} = y^2$ to argue:
$$e^{2i\pi\theta} \in \mathrm{Spec}(U) \implies e^{2i\pi\frac{\theta}{2}}, e^{2i\pi\left(\frac{\theta}{2}+\frac{1}{2}\right)} \in \mathrm{Spec}(U)$$
3. Iterate: spectrum size keeps doubling until $\epsilon$-errors kick in.

# Questions

How fast can hyperlinear profile grow?

(current best—$1/\epsilon^{2/3}$—won't last for long)

# Questions

How fast can hyperlinear profile grow?

(current best—$1/\epsilon^{2/3}$—won't last for long)

Can lower bounds on hyperlinear profile always be turned into lower bounds on entanglement?

(hard part: dimension-dependent factors crop up when going from max-entangled states to general states)

# Other consequences of embedding theorems

[Slofstra '16] since every f.-p. group embeds in a solution group...

word problem for groups is undecidable
$\implies$ undecidable to determine if $\omega_{qc}(G) = 1$

# Other consequences of embedding theorems

[Slofstra '16] since every f.-p. group embeds in a solution group...

word problem for groups is undecidable
$\implies$ undecidable to determine if $\omega_{qc}(G) = 1$

**Theorem**

*Some finitely-presented groups $K$ embed in solution groups $\Gamma$, in such a way that approximate representations of $K$ lift to approximate representations of $\Gamma$.*

## Other consequences of embedding theorems

[Slofstra '16] since every f.-p. group embeds in a solution group...

word problem for groups is undecidable
$\implies$ undecidable to determine if $\omega_{qc}(G) = 1$

**Theorem**

*Some finitely-presented groups $K$ embed in solution groups $\Gamma$, in such a way that approximate representations of $K$ lift to approximate representations of $\Gamma$.*

Using groups of Kharlampovich, Kharlampovich-Myasnikov-Sapir:

**Theorem (Slofstra)**

*For linear system games $G$, it is undecidable to determine if $\omega_q(G) = 1$ or if $G$ has a perfect finite-dimensional strategy.*

## Concluding remarks

Given $G$ and $\epsilon > 0$, can we compute $\omega_q(G) \pm \epsilon$?

If Connes' Embedding Conjecture is true: *yes!*

Effective bounds seem inherently tied to dimension bounds.

## Concluding remarks

### Given $G$ and $\epsilon > 0$, can we compute $\omega_q(G) \pm \epsilon$?

If Connes' Embedding Conjecture is true: *yes!*

Effective bounds seem inherently tied to dimension bounds.

### A "fully explicit" embedding theorem?

Remove dependence on black-box group-theoretic reductions

[Regev-V.'18] A *correlation* that "self-tests" the algebraic structure of any subset of an algebra

## Concluding remarks

### Given $G$ and $\epsilon > 0$, can we compute $\omega_q(G) \pm \epsilon$?

If Connes' Embedding Conjecture is true: *yes!*

Effective bounds seem inherently tied to dimension bounds.

### A "fully explicit" embedding theorem?

Remove dependence on black-box group-theoretic reductions

[Regev-V.'18] A *correlation* that "self-tests" the algebraic structure of any subset of an algebra

What are the interesting groups?

Applications to complexity. Crypto?