

## Tsirelson bounds for generalized Clauser-Horne-Shimony-Holt inequalities

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Quantum theory imposes a strict limit on the strength of nonlocal correlations. It only allows for a violation of the Clauser, Horne, Shimony, and Holt (CHSH) inequality up to the value  $2\sqrt{2}$ , known as Tsirelson's bound. In this paper, we consider generalized CHSH inequalities based on many measurement settings with two possible measurement outcomes each. We demonstrate how to prove Tsirelson bounds for any such generalized CHSH inequality using semidefinite programming. As an example, we show that for any shared entangled state and observables  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_n$  with eigenvalues  $\pm 1$  we have  $|\langle X_1 Y_1 \rangle + \langle X_2 Y_1 \rangle + \langle X_2 Y_2 \rangle + \langle X_3 Y_2 \rangle + \dots + \langle X_n Y_n \rangle - \langle X_1 Y_n \rangle| \leq 2n \cos[\pi/(2n)]$ . It is well known that there exist observables such that equality can be achieved. However, we show that these are indeed optimal. Our approach can easily be generalized to other inequalities for such observables.

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Nonlocal correlations arise as the result of measurements performed on a quantum system shared between two spatially separated parties. Imagine two parties, Alice and Bob, who are given access to a shared quantum state  $|\Psi\rangle$ , but cannot communicate. In the simplest case, each of them is able to perform one of two possible measurements. Every measurement has two possible outcomes labeled  $\pm 1$ . Alice and Bob now measure  $|\Psi\rangle$  using an independently chosen measurement setting and record their outcomes. In order to obtain an accurate estimate for the correlation between their measurement settings and the measurement outcomes, they perform this experiment many times using an identically prepared state  $|\Psi\rangle$  in each round. Both classical and quantum theories impose limits on the strength of such nonlocal correlations. In particular, both do not violate the nonsignaling condition of special relativity. That is, the local choice of measurement setting does not allow Alice and Bob to transmit information. Limits on the strength of correlations which are possible in the framework of any classical theory, i.e., a framework based on local hidden variables, are known as *Bell inequalities* [1]. The best known Bell inequality is the Clauser, Horne, Shimony, and Holt (CHSH) inequality [2]

$$|\langle X_1 Y_1 \rangle + \langle X_1 Y_2 \rangle + \langle X_2 Y_1 \rangle - \langle X_2 Y_2 \rangle| \leq 2,$$

where  $\{X_1, X_2\}$  and  $\{Y_1, Y_2\}$  are the observables representing the measurement settings of Alice and Bob respectively.  $\langle X_i Y_j \rangle = \langle \Psi | X_i \otimes Y_j | \Psi \rangle$  denotes the mean value of  $X_i$  and  $Y_j$ . Quantum mechanics allows for a violation of the CHSH inequality, but curiously still limits the strength of nonlocal correlations. Tsirelson's bound [3] says that for quantum mechanics

$$|\langle X_1 Y_1 \rangle + \langle X_1 Y_2 \rangle + \langle X_2 Y_1 \rangle - \langle X_2 Y_2 \rangle| \leq 2\sqrt{2}.$$

Peres demonstrated how to derive Bell inequalities [4] even for more than two settings. As Froissart and Tsirelson [5] have shown, these inequalities correspond to the faces of

a polytope. Computing the boundary of the space of correlations that can be attained using a classical theory therefore corresponds to determining the faces of this polytope. However, determining bounds on the correlations that *quantum* theory allows remains an even more difficult problem [6]. All Tsirelson's bounds are known for CHSH-type inequalities (also known as correlation inequalities) with two measurement settings and two outcomes for both Alice and Bob [5]. Filipp and Svozil [7] have considered the case of three measurement settings analytically and conducted numerical studies for a larger number of settings. Finally, Buhrman and Massar have shown a bound for a generalized CHSH inequality using three measurement settings with three outcomes each [6].

In this paper, we investigate the case where Alice and Bob can choose from  $n$  measurement settings with two outcomes each. We use a completely different approach based on semidefinite programming in combination with Tsirelson's seminal results [3,5,8]. This method is similar to methods used in computer science for the two-way partitioning problem [9] and the approximation algorithm for MAXCUT by Goemans and Williamson [10]. Cleve *et al.* [11] have also remarked that Tsirelson's constructions leads to an approach by semidefinite programming in the context of multiple interactive proof systems with entanglement. Semidefinite programming allows for an efficient way to approximate Tsirelson's bounds for any CHSH-type inequalities numerically. However, it can also be used to prove Tsirelson type bounds analytically. As an illustration, we first give an alternative proof of Tsirelson's original bound using semidefinite programming. We then prove a new Tsirelson's bound for the following generalized CHSH inequality [12,13]. Classically, it can be shown that

$$\left| \sum_{i=1}^n \langle X_i Y_i \rangle + \sum_{i=1}^{n-1} \langle X_{i+1} Y_i \rangle - \langle X_1 Y_n \rangle \right| \leq 2n - 2.$$

Here, we show that for quantum mechanics

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$$\left| \sum_{i=1}^n \langle X_i Y_i \rangle + \sum_{i=1}^{n-1} \langle X_{i+1} Y_i \rangle - \langle X_1 Y_n \rangle \right| \leq 2n \cos\left(\frac{\pi}{2n}\right),$$

where  $\{X_1, \dots, X_n\}$  and  $\{Y_1, \dots, Y_n\}$  are observables with eigenvalues  $\pm 1$  employed by Alice and Bob respectively, corresponding to their  $n$  possible measurement settings. It is well known that this bound can be achieved [12,13] for a specific set of measurement settings if Alice and Bob share a singlet state. Here, we show that this bound is indeed optimal for *any* state  $|\Psi\rangle$  and choice of measurement settings. This method generalizes to other CHSH inequalities, for example, the inequality considered by Gisin [14]. As outlined below, Tsirelson's results also imply that any bound proved using this method can indeed be achieved using quantum mechanics. As Braunstein and Caves [13] have shown, it is interesting to consider inequalities based on many measurement settings, in particular, the chained CHSH inequality above. The gap between the classical and the quantum bound for this inequality is larger than for the original CHSH inequality with only two measurement settings. They show that even for real experiments that inevitably include noise, this inequality leads to a stronger violation of local realism, and may thus lead to a better test.

## I. PRELIMINARIES

Throughout this paper, we write  $u=(u_1, \dots, u_n)$  for an  $n$ -element vector.  $u \cdot v$  denotes the inner product between vectors  $u$  and  $v$ . Furthermore,  $\text{diag}(\lambda)$  denotes the matrix with the components of the vector  $\lambda$  on its diagonal. We write  $A=[a_{ij}]$  to indicate that the entry in the  $i$ th row and  $j$ th column of  $A$  is  $a_{ij}$ . We also use the shorthand  $[n]=\{1, \dots, n\}$ .  $A^\dagger$  is the conjugate transpose of matrix  $A$ . A positive semidefinite  $n \times n$  matrix  $A$  is a nonsingular Hermitian matrix such that  $x^* A x \geq 0$  for all  $x \in \mathbb{C}^n$  [15]. We use  $A \geq 0$  to indicate that  $A$  is positive semidefinite. It will be important that a Hermitian matrix  $A$  is positive semidefinite if and only if all of its eigenvalues are nonnegative ([15], Theorem 7.2.1).

We will need two ingredients for our proof. First, the following result by Tsirelson ([3], Theorem 1) and [5,8] plays an essential role.

*Theorem 1 (Tsirelson).* Let  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_n$  be observables with eigenvalues in the interval  $[-1; 1]$ . Then for any state  $|\Psi\rangle \in \mathcal{A} \otimes \mathcal{B}$  and for all  $s, t \in [n]$  there exist real unit vectors  $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}^{2^n}$  such that

$$\langle \Psi | X_s \otimes Y_t | \Psi \rangle = x_s \cdot y_t.$$

Conversely, let  $x_s, y_t \in \mathbb{R}^N$  be real unit vectors. Let  $|\Psi\rangle \in \mathcal{A} \otimes \mathcal{B}$  be any maximally entangled state where  $\dim(\mathcal{A}) = \dim(\mathcal{B}) = 2^{\lfloor N/2 \rfloor}$ . Then for all  $s, t \in [n]$  there exist observables  $X_s$  on  $\mathcal{A}$  and  $Y_t$  on  $\mathcal{B}$  with eigenvalues in  $[-1; 1]$  such that

$$x_s \cdot y_t = \langle \Psi | X_s \otimes Y_t | \Psi \rangle.$$

In particular, this means that we can rewrite CHSH inequalities in terms of vectors. The second part of Tsirelson's result implies that any strategy based on vectors can indeed be implemented using quantum measurements. See [5] for a detailed construction.

Second, we will make use of semidefinite programming. This is a special case of convex optimization. We refer to [9] for an in-depth introduction. The goal of semidefinite programming is to solve the following semidefinite program (SDP) in terms of the variable  $X \in \mathcal{S}^n$

$$\text{maximize } \text{Tr}(CX),$$

$$\text{subject to } \text{Tr}(A_i X) = b_i, \quad i = 1, \dots, p, \quad \text{and } X \geq 0,$$

for given matrices  $C, A_1, \dots, A_p \in \mathcal{S}^n$  where  $\mathcal{S}^n$  is the space of symmetric  $n \times n$  matrices.  $X$  is called *feasible*, if it satisfies all constraints. An important aspect of semidefinite programming is duality. Intuitively, the idea behind Lagrangian duality is to extend the objective function [here  $\text{Tr}(CX)$ ] with a weighted sum of the constraints in such a way, that we will be penalized if the constraints are not fulfilled. The weights then correspond to the dual variables. Optimizing over these weights then gives rise to the *dual problem*. The original problem is called the *primal problem*. An example of this approach is given in the next section. Let  $d'$  denote the optimal value of the dual problem, and  $p'$  the optimal value of the primal problem from above. Weak duality says that  $d' \geq p'$ . In particular, if we have  $d' = p'$  for a feasible dual and primal solution respectively, we can conclude that both solutions are optimal.

## II. TSIRELSON'S BOUND

To illustrate our approach we first give a detailed proof of Tsirelson's bound using semidefinite programming. This proof is more complicated than Tsirelson's original proof, however, it serves as a good introduction to the following section. Let  $X_1, X_2$  and  $Y_1, Y_2$  denote the observables with eigenvalues  $\pm 1$  used by Alice and Bob, respectively. Our goal is now to show an upper bound for

$$|\langle X_1 Y_1 \rangle + \langle X_1 Y_2 \rangle + \langle X_2 Y_1 \rangle - \langle X_2 Y_2 \rangle|.$$

From Theorem 1 we know that there exist real unit vectors  $x_s, y_t \in \mathbb{R}^4$  such that for all  $s, t \in \{0, 1\}$   $\langle X_s Y_t \rangle = x_s \cdot y_t$ . In order to find Tsirelson's bound, we thus want to solve the following problem: maximize  $x_1 \cdot y_1 + x_1 \cdot y_2 + x_2 \cdot y_1 - x_2 \cdot y_2$ , subject to  $\|x_1\| = \|x_2\| = \|y_1\| = \|y_2\| = 1$ . Note that we can drop the absolute value since any set of vectors maximizing the above equation, simultaneously leads to a set of vectors minimizing it by taking  $-y_1, -y_2$  instead. We will now phrase this as a semidefinite program. Let  $G = [g_{ij}]$  be the Gram matrix of the vectors  $\{x_1, x_2, y_1, y_2\} \subseteq \mathbb{R}^4$  with respect to the inner product

$$G = \begin{pmatrix} x_1 \cdot x_1 & x_1 \cdot x_2 & x_1 \cdot y_1 & x_1 \cdot y_2 \\ x_2 \cdot x_1 & x_2 \cdot x_2 & x_2 \cdot y_1 & x_2 \cdot y_2 \\ y_1 \cdot x_1 & y_1 \cdot x_2 & y_1 \cdot y_1 & y_1 \cdot y_2 \\ y_2 \cdot x_1 & y_2 \cdot x_2 & y_2 \cdot y_1 & y_2 \cdot y_2 \end{pmatrix}.$$

$G$  can thus be written as  $G = B^T B$  where the columns of  $B$  are the vectors  $\{x_1, x_2, y_1, y_2\}$ . By ([15], Theorem 7.2.11) we can write  $G = B^T B$  if and only if  $G$  is positive semidefinite. We thus impose the constraint that  $G \geq 0$ . To make sure that we obtain unit vectors, we add the constraint that all diagonal entries of  $G$  must be equal to 1. Define

$$W = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

Note that the choice of order of the vectors in  $B$  is not unique, however, a different order only leads to a different  $W$  and does not change our argument. We can now rephrase our optimization problem as the following SDP:

$$\text{maximize } \frac{1}{2} \text{Tr}(GW),$$

$$\text{subject to } G \geq 0 \text{ and } \forall i, g_{ii} = 1.$$

We can then write for the Lagrangian

$$L(G, \lambda) = \frac{1}{2} \text{Tr}(GW) - \text{Tr}[\text{diag}(\lambda)(G - I)],$$

where  $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ . The dual function is then

$$g(\lambda) = \sup_G \text{Tr}\left\{G\left[\frac{1}{2}W - \text{diag}(\lambda)\right]\right\} + \text{Tr}[\text{diag}(\lambda)] \\ = \begin{cases} \text{Tr}[\text{diag}(\lambda)] & \text{if } \frac{1}{2}W - \text{diag}(\lambda) \leq 0 \\ \infty & \text{otherwise.} \end{cases}$$

We then obtain the following dual formulation of the SDP

$$\text{minimize } \text{Tr}[\text{diag}(\lambda)],$$

$$\text{subject to } -\frac{1}{2}W + \text{diag}(\lambda) \geq 0.$$

Let  $p'$  and  $d'$  denote optimal values for the primal and Lagrange dual problem respectively. From weak duality it follows that  $d' \geq p'$ . For our example, it is not difficult to see that this is indeed true. Let  $G'$  and  $\lambda'$  be optimal solutions of the primal and dual problem, i.e.,  $p' = \frac{1}{2} \text{Tr}(G'W)$  and  $d' = \text{Tr}[\text{diag}(\lambda')]$ . Recall that all entries on the diagonal of  $G'$  are 1 and thus  $\text{Tr}[\text{diag}(\lambda')] = \text{Tr}[G' \text{diag}(\lambda')]$ . Then

$$d' - p' = \text{Tr}[\text{diag}(\lambda')] - \frac{1}{2} \text{Tr}(G'W) = \text{Tr}\left[G' \text{diag}(\lambda') - \frac{1}{2}G'W\right] = \text{Tr}\left\{G'\left[\text{diag}(\lambda') - \frac{1}{2}W\right]\right\} \geq 0,$$

where the last inequality follows from the constraints  $G' \geq 0$ ,  $\text{diag}(\lambda') - \frac{1}{2}W \geq 0$  and [9, Example 2.24].

In order to prove Tsirelson's bound, we will now exhibit an optimal solution for both the primal and dual problem and then show that the value of the primal problem equals the value of the dual problem. The optimal solution is well known [3,8,12]. Alternatively, we could easily guess the optimal solution based on numerical optimization by a small program for Matlab [18] and the package SeDuMi [16] for semidefinite programming. Consider the following solution for the primal problem

$$G' = \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix},$$

which gives rise to the primal value  $p' = \frac{1}{2} \text{Tr}(G'W) = 2\sqrt{2}$ . Note that  $G' \geq 0$  since all its eigenvalues are nonnegative [15, Theorem 7.2.1] and all its diagonal entries are 1. Thus all constraints are satisfied. The lower left quadrant of  $G'$  is in fact the same as the well known correlation matrix for 2 observables ([5], Equation 3.16). Next, consider the following solution for the dual problem

$$\lambda' = \frac{1}{\sqrt{2}}(1, 1, 1, 1).$$

The dual value is then  $d' = \text{Tr}[\text{diag}(\lambda')] = 2\sqrt{2}$ . Because  $-\text{diag}(\lambda') + \frac{1}{2}W \geq 0$ ,  $\lambda'$  satisfies the constraint. Since  $p' = d'$ ,  $G'$  and  $\lambda'$  are in fact optimal solutions for the primal and dual respectively. We can thus conclude that

$$|\langle X_1 Y_1 \rangle + \langle X_1 Y_2 \rangle + \langle X_2 Y_1 \rangle - \langle X_2 Y_2 \rangle| \leq 2\sqrt{2},$$

which is Tsirelson's bound [3]. By Theorem 1, this bound is achievable.

### III. TSIRELSON'S BOUNDS FOR MORE THAN TWO OBSERVABLES

We now show how to obtain bounds for inequalities based on more than two observables for both Alice and Bob. In particular, we will prove a bound for the chained CHSH inequality for the quantum case. It is well known [12] that it is possible to choose observables  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_n$  such that

$$\left| \sum_{i=1}^n \langle X_i Y_i \rangle + \sum_{i=1}^{n-1} \langle X_{i+1} Y_i \rangle - \langle X_1 Y_n \rangle \right| = 2n \cos\left(\frac{\pi}{2n}\right).$$

We now show that this is optimal. Our proof is similar to the last section. However, it is more difficult to show feasibility for all  $n$ .

*Theorem 2.* Let  $\rho \in \mathcal{A} \otimes \mathcal{B}$  be an arbitrary state, where  $\mathcal{A}$  and  $\mathcal{B}$  denote the Hilbert spaces of Alice and Bob. Let  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_n$  be observables with eigenvalues  $\pm 1$  on  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. Then

$$\left| \sum_{i=1}^n \langle X_i Y_i \rangle + \sum_{i=1}^{n-1} \langle X_{i+1} Y_i \rangle - \langle X_1 Y_n \rangle \right| \leq 2n \cos\left(\frac{\pi}{2n}\right).$$

*Proof.* By Theorem 1, our goal is to find the maximum value for  $x_1 \cdot y_1 + x_2 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_2 + \dots + x_n \cdot y_n - x_1 \cdot y_n$ , for real unit vectors  $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}^{2n}$ . As above we can drop the absolute value. Let  $G = [g_{ij}]$  be the Gram matrix of

the vectors  $\{x_1, \dots, x_n, y_1, \dots, y_n\} \subseteq \mathbb{R}^{2n}$ . As before, we can thus write  $G = B^T B$ , where the columns of  $B$  are the vectors  $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ , if and only if  $G \geq 0$ . To ensure we obtain unit vectors, we again demand that all diagonal entries of  $G$  equal 1. Define  $n \times n$  matrix  $A$  and  $2n \times 2n$  matrix  $W$  by

$$A = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 1 & 1 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}.$$

We can now phrase our maximization problem as the following SDP:

$$\begin{aligned} & \text{maximize } \frac{1}{2} \text{Tr}(GW) \\ & \text{subject to } G \geq 0 \text{ and } \forall i, g_{ii} = 1. \end{aligned}$$

Analog to the previous section, the dual SDP is then

$$\begin{aligned} & \text{minimize } \text{Tr}[\text{diag}(\lambda)], \\ & \text{subject to } -\frac{1}{2}W + \text{diag}(\lambda) \geq 0. \end{aligned}$$

Let  $p'$  and  $d'$  denote optimal values for the primal and dual problem respectively. As before,  $d' \geq p'$ .

*a Primal.* We will now show that the vectors suggested in [12] are optimal. For  $k \in [n]$ , choose unit vectors  $x_k, y_k \in \mathbb{R}^{2n}$  to be of the form

$$\begin{aligned} x_k &= [\cos(\phi_k), \sin(\phi_k), 0, \dots, 0], \\ y_k &= [\cos(\psi_k), \sin(\psi_k), 0, \dots, 0], \end{aligned}$$

where  $\phi_k = (\pi/(2n))(2k-2)$  and  $\psi_k = (\pi/(2n))(2k-1)$ . The angle between  $x_k$  and  $y_k$  is given by  $\psi_k - \phi_k = \pi/(2n)$  and thus  $x_k \cdot y_k = \cos(\pi/(2n))$ . The angle between  $x_{k+1}$  and  $y_k$  is  $\phi_{k+1} - \psi_k = \pi/(2n)$  and thus  $x_{k+1} \cdot y_k = \cos(\pi/(2n))$ . Finally, the angle between  $-x_1$  and  $y_n$  is  $\pi - \psi_n = \pi/(2n)$  and so  $-x_1 \cdot y_n = \cos(\pi/(2n))$ . The value of our primal problem is thus given by

$$p' = \sum_{k=1}^n x_k \cdot y_k + \sum_{k=1}^{n-1} x_{k+1} \cdot y_k - x_1 \cdot y_n = 2n \cos\left(\frac{\pi}{2n}\right).$$

Let  $G'$  be the Gram matrix constructed from all vectors  $x_k, y_k$  as described earlier. Note that our constraints are satisfied:  $\forall i: g_{ii} = 1$  and  $G' \geq 0$ , because  $G'$  is symmetric and of the form  $G' = B'^T B'$ .

*b Dual.* Now consider the  $n$ -dimensional vector

$$\lambda' = \cos\left(\frac{\pi}{2n}\right)(1, \dots, 1).$$

In order to show that this is a feasible solution to the dual problem, we have to prove that  $-\frac{1}{2}W + \text{diag}(\lambda') \geq 0$  and thus the constraint is satisfied. To this end, we first show that

*Claim 1.* The eigenvalues of  $A$  are given by  $\gamma_s = 1 + e^{i\pi(2s+1)/n}$  with  $s=0, \dots, n-1$ .

*Proof.* Note that if the lower left corner of  $A$  were 1,  $A$

would be a circulant matrix [17], i.e., each row of  $A$  is constructed by taking the previous row and shifting it one place to the right. We can use ideas from circulant matrices to guess eigenvalues  $\gamma_s$  with eigenvectors

$$u_s = (\rho_s^{n-1}, \rho_s^{n-2}, \rho_s^{n-3}, \dots, \rho_s^1, \rho_s^0),$$

where  $\rho_s = e^{-i\pi(2s+1)/n}$  and  $s=0, \dots, n-1$ . By definition,  $u = (u_1, u_2, \dots, u_n)$  is an eigenvector of  $A$  with eigenvalue  $\gamma$  if and only if  $Au = \gamma u$ . Here,  $Au = \gamma u$  if and only if

$$\begin{aligned} \text{(i)} \quad & \forall j \in \{1, \dots, n-1\}: u_j + u_{j+1} = \gamma u_j, \\ \text{(ii)} \quad & -u_1 + u_n = \gamma u_n. \end{aligned}$$

Since for any  $j \in \{1, \dots, n-1\}$

$$\begin{aligned} u_j + u_{j+1} &= \rho_s^{n-j} + \rho_s^{n-j-1} = e^{-i(n-j)\pi(2s+1)/n} (1 + e^{i\pi(2s+1)/n}) \\ &= \rho_s^{n-j} \gamma_s = \gamma_s u_j, \end{aligned}$$

(i) is satisfied. Furthermore (ii) is satisfied, since

$$\begin{aligned} -u_1 + u_n &= -\rho_s^{n-1} + \rho_s^0 = -e^{-i\pi(2s+1)} e^{i\pi(2s+1)/n} + 1 = 1 \\ &+ e^{i\pi(2s+1)/n} = \gamma_s \rho_s^0 = \gamma_s u_n. \end{aligned}$$

*Claim 2.* The largest eigenvalue of  $W$  is given by  $\gamma = 2 \cos(\pi/(2n))$ .

*Proof.* By [15, Theorem 7.3.7] the eigenvalues of  $W$  are given by the singular values of  $A$  and their negatives. It follows from Claim 1 that the singular values of  $A$  are

$$\sigma_s = \sqrt{\gamma_s \gamma_s^*} = 2 + 2 \cos\left(\frac{\pi(2s+1)}{n}\right).$$

Considering the shape of the cosine function, it is easy to see that the largest singular value of  $A$  is given by  $2 + 2 \cos(\pi/n) = 4 \cos^2[\pi/(2n)]$ , the largest eigenvalue of  $W$  is  $\sqrt{2 + 2 \cos(\pi/n)} = 2 \cos[\pi/(2n)]$ . ■

Since  $-\frac{1}{2}W + \text{diag}(\lambda')$  are both Hermitian, Weyl's theorem ([15], Theorem 4.3.1) implies that

$$\gamma_{\min}\left[-\frac{1}{2}W + \text{diag}(\lambda')\right] \geq \gamma_{\min}\left(-\frac{1}{2}W\right) + \gamma_{\min}[\text{diag}(\lambda')],$$

where  $\gamma_{\min}(M)$  is the smallest eigenvalue of a matrix  $M$ . It then follows from the fact that  $\text{diag}(\lambda')$  is diagonal and Claim 2 that

$$\gamma_{\min}\left[-\frac{1}{2}W + \text{diag}(\lambda')\right] \geq -\frac{1}{2}\left[2 \cos\left(\frac{\pi}{2n}\right)\right] + \cos\left(\frac{\pi}{2n}\right) = 0.$$

Thus  $-\frac{1}{2}W + \text{diag}(\lambda') \geq 0$  and  $\lambda'$  is a feasible solution to the dual problem. The value of the dual problem is then

$$d' = \text{Tr}[\text{diag}(\lambda')] = 2n \cos\left(\frac{\pi}{2n}\right).$$

Because  $p' = d'$ ,  $G'$  and  $\lambda'$  are optimal solutions for the primal and dual respectively, which completes our proof. ■

Note that for the primal problem we are effectively dealing with two-dimensional vectors,  $x_k, y_k$ . It therefore follows from Tsirelson's construction [5] that given an Einstein-Podolsky-Rosen (EPR) pair we can find observables such that the bound is tight. In fact, these vectors just determine the measurement directions as given in [12].

#### IV. DISCUSSION

Our approach can be generalized to other CHSH-type inequalities. For another inequality, we merely use a different matrix  $A$  in  $W$ . For example, for Gisin's CHSH inequality [14],  $A$  is the matrix with 1's in the upper left half and on the diagonal, and  $-1$ 's in the lower right part. Otherwise our approach stays exactly the same, and thus we do not consider this case here. Numerical results provided by our Matlab example code suggest that Gisin's observables are optimal. Given the framework of semidefinite programming, the only difficulty in proving bounds for other inequalities is to determine the eigenvalues of the corresponding  $A$ , a simple ma-

trix. Since our approach is based on Tsirelson's vector construction, it is limited to CHSH-type inequalities for two parties.

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