

Security of Quantum Bit String Commitment Depends on the Information Measure

Harry Buhrman,¹ Matthias Christandl,^{2,*} Patrick Hayden,³ Hoi-Kwong Lo,⁴ and Stephanie Wehner^{1,†}

¹*CWI, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands*

²*DAMTP, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, United Kingdom*

³*School of Computer Science, McGill University, Montreal, Canada*

⁴*Department of ECE and Physics, University of Toronto, Canada M5G 3G4*

(Received 12 October 2005; revised manuscript received 19 May 2006; published 19 December 2006)

Unconditionally secure nonrelativistic bit commitment is known to be impossible in both the classical and the quantum world. However, when committing to a string of n bits at once, how far can we stretch the quantum limits? In this Letter, we introduce a framework of quantum schemes where Alice commits a string of n bits to Bob, in such a way that she can only cheat on a bits and Bob can learn at most b bits of information before the reveal phase. Our results are twofold: we show by an explicit construction that in the traditional approach, where the reveal and guess probabilities form the security criteria, no good schemes can exist: $a + b$ is at least n . If, however, we use a more liberal criterion of security, the accessible information, we construct schemes where $a = 4\log_2 n + O(1)$ and $b = 4$, which is impossible classically. Our findings significantly extend known no-go results for quantum bit commitment.

DOI: [10.1103/PhysRevLett.97.250501](https://doi.org/10.1103/PhysRevLett.97.250501)

PACS numbers: 03.67.Dd

Imagine two mutually distrustful parties Alice and Bob at distant locations. They can only communicate over a channel, but want to play the following game: Alice secretly chooses a bit x . Bob wants to be sure that Alice indeed has made her choice. Yet, Alice wants to keep x hidden from Bob until she decides to reveal x . To convince Bob that she made up her mind, Alice sends Bob a commitment. From the commitment alone, Bob cannot deduce x . At a later time, Alice reveals x and enables Bob to open the commitment. Bob can now check if Alice is telling the truth. This scenario is known as *bit commitment*. Commitments play a central role in modern day cryptography. They form an important building block in the construction of larger protocols in, for example, gambling and electronic voting, and other instances of secure two-party computation. In the realm of quantum mechanics, it has been shown that oblivious transfer [1] can be achieved provided there exists a secure bit-commitment scheme [2,3]. In turn, classical oblivious transfer can be used to perform any secure two-party computation [4]. Commitments are also useful for constructing zero-knowledge proofs [5] and lead to coin tossing [6].

Classically, unconditionally secure nonrelativistic bit commitment is known to be impossible. Unfortunately, after several quantum schemes were suggested [7], nonrelativistic *quantum* bit commitment was shown to be impossible, too [8,9], even in the presence of superselection rules [10]. In fact, only very limited degrees of concealment and bindingness can be obtained [11]. It has been shown that the quantum no-go theorems do not apply to protocols which use two or more sites and take account of relativistic signaling constraints. In the face of the negative results, what can we still hope to achieve?

In this Letter, we consider the task of committing to a string of n bits at once when both the honest player and the adversary have unbounded resources. Since perfect bit

commitment is impossible, perfect bit string commitment is impossible, too. We thus give both Alice and Bob a limited ability to cheat. First, we introduce a framework for the classification of bit string commitments in terms of the length n of the string, Alice's ability to cheat on at most a bits, and Bob's ability to acquire at most b bits of information before the reveal phase. We say that Alice can cheat on a bits if she can reveal up to 2^a strings successfully. Bob's security definition is crucial to our investigation: if b determines a bound on his probability to guess Alice's string, then we prove that $a + b$ is at least n . This implies that the trivial protocol, where Alice's commitment consists of sending b bits of her string to Bob, is optimal. If, however, b is a bound on the accessible information that the quantum states contain about Alice's string, then we show that nontrivial schemes exist. More precisely, we construct schemes with $a = 4\log_2 n + O(1)$ and $b = 4$. This is impossible classically.

Quantum commitments of strings have previously been considered by Kent [12], who pointed out that in the quantum world useful bit string commitments could be possible despite the no-go theorem for bit commitment.

Framework. — *Definition 1.* — An (n, a, b) -quantum bit string commitment (QBSC) is a quantum communication protocol between two parties, Alice (the committer) and Bob (the receiver), which consists of two phases and two security requirements. (i) (Commit Phase) Assume that both parties are honest. Alice chooses a string $x \in \{0, 1\}^n$ with probability p_x . Alice and Bob communicate and at the end Bob holds state ρ_x . (ii) (Reveal Phase) If both parties are honest, Alice and Bob communicate and at the end Bob learns x . Bob accepts. (iii) (Concealing) If Alice is honest, $\sum_{x \in \{0,1\}^n} p_{x|x}^B \leq 2^b$, where $p_{x|x}^B$ is the probability that Bob correctly guesses x before the reveal phase. (iv) (Binding) If Bob is honest, then for all commitments of Alice:

$\sum_{x \in \{0,1\}^n} p_x^A \leq 2^a$, where p_x^A is the probability that Alice successfully reveals x .

We say that Alice *successfully reveals* a string x if Bob accepts the opening of x , i.e., he performs a test depending on the individual protocol to check Alice's honesty and concludes that she was indeed honest. Note that quantumly, Alice can always commit to a superposition of different strings without being detected. Thus even for a perfectly binding bit string commitment (i.e., $a = 0$) we only demand that $\sum_{x \in \{0,1\}^n} p_x^A \leq 1$, whereas classically one wants that $p_x^A = \delta_{x,x'}$. Note that our concealing definition reflects Bob's *a priori* knowledge about x . We choose an *a priori* uniform distribution (i.e., $p_x = 2^{-n}$) for (n, a, b) -QBSCs, which naturally comes from the fact that we consider n -bit strings. A generalization to any (P_X, a, b) -QBSC where P_X is an arbitrary distribution is possible but omitted in order not to obscure our main line of argument. Instead of Bob's guessing probability, one can take any information measure B to express the security against Bob. In general, we consider an (n, a, b) -QBSC $_B$ where the new concealing condition $B(\mathcal{E}) \leq b$ holds for any ensemble $\mathcal{E} = \{p_x, \rho_x\}$ that Bob can obtain by a cheating strategy. In the latter part of this Letter we show that for B being the *accessible information* nontrivial protocols, i.e., protocols with $a + b \ll n$, exist. The accessible information is defined as $I_{\text{acc}}(\mathcal{E}) = \max_M I(X; Y)$, where P_X is the prior distribution of the random variable X , Y is the random variable of the outcome of Bob's measurement on \mathcal{E} , and the maximization is taken over all measurements M .

Impossibility.—Our impossibility result will be proven in three steps: we first show that any (n, a, b) -QBSC is also an (n, a, b) -QBSC $_\xi$ with the security measure $\xi(\mathcal{E})$ defined below in Eq. (1). Second, we prove that an (n, a, b) -QBSC $_\xi$ can only exist for values a, b , and n obeying $a + b + c \geq n$, where c is a small constant independent of a, b , and n . This in turn implies the impossibility of an (n, a, b) -QBSC for such parameters. Finally, we show that *many* executions of the protocol can only be secure if $a + b \geq n$. Before we proceed to the proof, we introduce a few tools from quantum information theory.

We work in the model of two-party nonrelativistic quantum protocols of Yao [3], simplified by Lo and Chau [8] which is usually adopted in this context. Here, any two-party quantum protocol can be regarded as a pair of quantum machines (Alice and Bob), interacting through a quantum channel. Consider the product of three Hilbert spaces \mathcal{H}_A , \mathcal{H}_B , and \mathcal{H}_C of bounded dimensions representing the Hilbert spaces of Alice's and Bob's machines and the channel, respectively. Without loss of generality, we assume that each machine is initially in a specified pure state. Alice and Bob perform a number of rounds of communication over the channel. Each such round can be modeled as a unitary transformation on $\mathcal{H}_A \otimes \mathcal{H}_C$ and $\mathcal{H}_B \otimes \mathcal{H}_C$, respectively. Since the protocol is known to both Alice and Bob, they know the set of possible unitary transformations used in the protocol. We assume that Alice and

Bob are in possession of both a quantum computer and a quantum storage device. This enables them to add ancillae to the quantum machine and use reversible unitary operations to replace measurements. By doing so, Alice and Bob can delay measurements and thus we can limit ourselves to protocols where both parties only measure at the very end. Moreover, any classical computation or communication that may occur can be simulated by a quantum computer.

We now show that every (n, a, b) -QBSC is an (n, a, b) -QBSC $_\xi$. The security measure $\xi(\mathcal{E})$ is defined by

$$\xi(\mathcal{E}) \equiv n - H_2(\rho_{AB}|\rho), \quad (1)$$

where $\rho_{AB} = \sum_x p_x |x\rangle\langle x| \otimes \rho_x$ and $\rho = \sum_x p_x \rho_x$ are only dependent on the ensemble $\mathcal{E} = \{p_x, \rho_x\}$. $H_2(\cdot|\cdot)$ is an entropic quantity defined in [13] $H_2(\rho^{AB}|\rho) \equiv -\log_2 \text{Tr}[(\mathbb{1} \otimes \rho^{-(1/2)})\rho_{AB}]^2$. This quantity is directly connected to Bob's maximal average probability of successful guessing the string:

Lemma 1.—Bob's maximal average probability of successfully guessing the committed string, i.e., $\sup_M \sum_x p_x p_{x|x}^{B,M}$, where M ranges over all measurements and $p_{y|x}^{B,M}$ is the conditional probability of guessing y given ρ_x , is larger or equal to $2^{-H_2(\rho_{AB}|\rho)}$.

Proof.—By definition, the maximum average guessing probability is lower bounded by the average guessing probability for a particular measurement strategy. We choose the *square-root measurement* which has operators $M_x = p_x \rho^{-(1/2)} \rho_x \rho^{-(1/2)}$. $p_{x|x}^B = \text{Tr}(M_x \rho_x)$ is the probability that Bob guesses x given ρ_x , hence

$$\begin{aligned} \log_2 \sum_x p_x p_{x|x}^{B,\max} &\geq \log_2 \sum_x p_x^2 \text{Tr}(\rho^{-(1/2)} \rho_x \rho^{-(1/2)} \rho_x) \\ &= \log_2 \text{Tr}[(\mathbb{1} \otimes \rho^{-(1/2)})\rho_{AB}]^2 \\ &= -H_2(\rho_{AB}|\rho). \quad \square \end{aligned}$$

Related estimates were derived in [14]. For the uniform distribution $p_x = 2^{-n}$ we have from the concealing condition that $\sum_x p_x p_{x|x}^B \leq 2^b$ which by Lemma 1 implies $\xi(\mathcal{E}) \leq b$. Thus, an (n, a, b) -QBSC is an (n, a, b) -QBSC $_\xi$.

We make use of the following theorem, known as privacy amplification against a quantum adversary. In our case, Bob holds the quantum memory and privacy amplification is used to find Alice's attack.

Theorem 1 [Th. 5.5.1 in [13] (see also [15])]—Let \mathcal{G} be a class of two-universal hash functions from $\{0, 1\}^n$ to $\{0, 1\}^s$. Application of $g \in \mathcal{G}$ to the random variable X maps the ensemble $\mathcal{E} = \{p_x, \rho_x\}$ to $\mathcal{E}_g = \{q_y^g, \sigma_y^g\}$ with probabilities $q_y^g = \sum_{x \in g^{-1}(y)} p_x$ and states $\sigma_y^g = \sum_{x \in g^{-1}(y)} p_x \rho_x$. Then

$$\frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} d(\mathcal{E}_g) \leq \frac{1}{2} 2^{-(1/2)[H_2(\rho_{AB}|\rho) - s]}, \quad (2)$$

where $d(\mathcal{E}) \equiv \delta(\sum_x p_x |x\rangle\langle x| \otimes \rho_x, \mathbb{1}/2^n \otimes \rho)$ [and similarly for $d(\mathcal{E}_g)$] and $\delta(\alpha, \beta) \equiv \frac{1}{2} \text{Tr}|\alpha - \beta|$.

The following reasoning is used to prove the impossibility of quantum bit commitment [8,9]: suppose ρ_0 and ρ_1

are density operators that correspond to a commitment of a “0” or a “1,” respectively. Let $|\phi_0\rangle$ and $|\phi_1\rangle$ be the corresponding purifications on the joint system of Alice and Bob. If ρ_0 equals ρ_1 then Alice can find a local unitary transformation U that Alice can apply to her part of the system such that $|\phi_1\rangle = U \otimes \mathbb{1}|\phi_0\rangle$. This enables Alice to change the total state from $|\phi_0\rangle$ to $|\phi_1\rangle$ and thus cheat. This reasoning also holds in an approximate sense [9], here used in the following form.

Lemma 2.—Let $\delta(\rho_0, \rho_1) \leq \epsilon$ and assume that the bit-commitment protocol is error-free if both parties are honest. Then there is a method for Alice to cheat such that the probability of successfully revealing a 0 given that she committed to a 1 is greater or equal to $1 - \sqrt{2\epsilon}$.

Theorem 2.— (n, a, b) -QBSC $_\xi$ schemes, and thus also (n, a, b) -QBSC schemes, with $a + b + c < n$, do not exist. c is a constant equal to $5\log_2 5 - 4 \approx 7.61$.

Proof.—Consider an (n, a, b) -QBSC $_\xi$ and the case where both Alice and Bob are honest. Alice committed to x . We denote the joint state of the system Alice-Bob channel $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ after the commit phase by $|\phi_x\rangle$ for input state $|x\rangle$. Let ρ_x be Bob’s reduced density matrix, and let $\mathcal{E} = \{p_x, \rho_x\}$ where $p_x = 2^{-n}$.

Assuming that Bob is honest, we will give a cheating strategy for Alice in the case where $a + b + 5\log_2 5 - 4 < n$. The strategy will depend on the two-universal hash function $g: \mathcal{X} = \{0, 1\}^n \rightarrow \mathcal{Y} = \{0, 1\}^{n-m}$, for appropriately chosen m . Alice picks a $y \in \mathcal{Y}$ and constructs the state $(\sum_{x \in g^{-1}(y)} |x\rangle|x\rangle) / \sqrt{|g^{-1}(y)|}$. She then gives the second half of this state as input to the protocol and stays honest for the rest of the commit phase. The joint state of Alice and Bob at the end of the commit phase is thus $|\psi_y^g\rangle = (\sum_{x \in g^{-1}(y)} |x\rangle|\phi_x\rangle) / \sqrt{|g^{-1}(y)|}$. The reduced states on Bob’s side are $\sigma_y^g = \frac{1}{q_y^g} \sum_{x \in g^{-1}(y)} p_x \rho_x$ with probability $q_y^g = \sum_{x \in g^{-1}(y)} p_x$. We denote this ensemble by \mathcal{E}_g . Let $\sigma = \sigma^g = \sum_y q_y^g \sigma_y^g$ for all g .

We now apply Theorem 1 with $s = n - m$ and $\xi(\mathcal{E}) \leq b$ and obtain $\frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} d(\mathcal{E}_g) \leq \epsilon$, where $\epsilon = \frac{1}{2} 2^{-(1/2)(m-b)}$. Hence, there is at least one g such that $d(\mathcal{E}_g) \leq \epsilon$; intuitively, this means that Bob knows only very little about the value of $g(x)$. This g defines Alice’s cheating strategy. It is straightforward to verify that $d(\mathcal{E}_g) \leq \epsilon$ implies

$$2^{-(n-m)} \sum_y \delta(\sigma, \sigma_y^g) \leq 2\epsilon. \quad (3)$$

Let us therefore assume without loss of generality that Alice chooses $y_0 \in \mathcal{Y}$ with $\delta(\sigma_{y_0}^g, \sigma) \leq 2\epsilon$.

Clearly, the probability to successfully reveal some x in $g^{-1}(y)$ given $|\psi_y^g\rangle$ is one [16]. Thus the probability to reveal y [i.e., to reveal an x such that $y = g(x)$] given $|\psi_y^g\rangle$ successfully is one. Let \tilde{p}_x and \tilde{q}_y^g denote the probabilities to successfully reveal x and y , respectively, and $\tilde{p}_{x|y}^g$ be the conditional probability to successfully reveal x , given y . We have $\sum_x \tilde{p}_x = \sum_y \tilde{q}_y^g \sum_{x \in g^{-1}(y)} \tilde{p}_{x|y}^g \geq \sum_y \tilde{q}_y^g$.

Recall that Alice can transform $|\psi_{y_0}^g\rangle$ approximately into $|\psi_y^g\rangle$ if $\sigma_{y_0}^g$ is sufficiently close to σ_y^g by using only local transformations on her part. It follows from Lemma 2 that we can estimate the probability of revealing y , given that the state was really $|\psi_{y_0}^g\rangle$. Since this reasoning applies to all y , on average, we have

$$\begin{aligned} \sum_y \tilde{q}_y^g &\geq \sum_y [1 - 2^{1/2} \delta(\sigma_{y_0}^g, \sigma_y^g)^{1/2}] \\ &\geq 2^{n-m} - 2^{1/2} 2^{n-m} \left[2^{m-n} \sum_y \delta(\sigma_{y_0}^g, \sigma_y^g) \right]^{1/2} \\ &\geq 2^{n-m} \left[1 - 2^{1/2} \left(2^{m-n} \left[\sum_y \delta(\sigma_{y_0}^g, \sigma) \right. \right. \right. \\ &\quad \left. \left. \left. + \delta(\sigma, \sigma_y^g) \right] \right)^{1/2} \right] \\ &\geq 2^{n-m} [1 - 2(2\epsilon)^{1/2}], \end{aligned}$$

where the first inequality follows from Lemma 2, the second from Jensen’s inequality and the concavity of the square-root function, the third from the triangle inequality, and the fourth from Eq. (3) and $\delta(\sigma_{y_0}^g, \sigma) \leq 2\epsilon$. Recall that to be secure against Alice, we require $2^a \geq 2^{n-m} [1 - 2(2\epsilon)^{1/2}]$. We insert $\epsilon = \frac{1}{2} 2^{-(1/2)(m-b)}$, define $m = b + \gamma$, and take the logarithm on both sides to get

$$a + b + \delta \geq n, \quad (4)$$

where $\delta = \gamma - \log_2(1 - 2^{-\gamma/4+1})$. Keeping in mind that $1 - 2^{-\gamma/4+1} > 0$ (or equivalently $\gamma > 4$), we find that the minimum value of δ for which Eq. (4) is satisfied is $\delta = 5\log_2 5 - 4$ and arises from $\gamma = 4(\log_2 5 - 1)$. Thus, no (n, a, b) -QBSC $_\xi$ with $a + b + 5\log_2 5 - 4 < n$ exists. \square

Since the constant c does not depend on a, b , and n , multiple parallel executions of the protocol can only be secure if $a + b \geq n$:

Corollary 1.—Let P be an (n, a, b) -QBSC with P^m an (mn, ma, mb) -QBSC. Then $n < a + b + c/m$. In particular, no (n, a, b) -QBSC with $a + b < n$ can be executed securely an arbitrary number of times in parallel. The latter statement also applies to (n, a, b) -QBSC $_{\chi, s}$, where χ denotes the Holevo information of the ensemble \mathcal{E} [17].

It follows directly from [10] that the results in this section also hold in the presence of superselection rules.

Possibility.—Surprisingly, if one is willing to measure Bob’s ability to learn x using the accessible information, nontrivial protocols become possible. These protocols are based on a discovery known as “locking of classical information in quantum states” [18]. The protocol, which we call LOCKCOM (n, \mathcal{U}) , uses this effect and is specified by a set $\mathcal{U} = \{U_1, \dots, U_{|\mathcal{U}|}\}$ of unitaries. (i) Commit phase: Alice has the string $x \in \{0, 1\}^n$ and randomly chooses $r \in \{1, \dots, |\mathcal{U}|\}$. She sends the state $U_r|x\rangle$ to Bob, where $U_r \in \mathcal{U}$. (ii) Reveal phase: Alice announces r and x . Bob applies U_r^\dagger and measures in the computational basis to obtain x' . He accepts if and only if $x' = x$.

As a first observation, the number of unitaries $|\mathcal{U}|$ limits the number of different ways of revealing a string, i.e., $2^a \leq |\mathcal{U}|$ [19]. Furthermore, we have adapted the work in [20] in order to show that there exist $O(n^4)$ unitaries that bring Bob's accessible information down to a constant: $I_{acc}(\mathcal{E}) \leq 4$ ([17], Appendix B.2). In summary:

Theorem 3.—For $n \geq 3$, there exist $(n, 4\log_2 n + O(1), 4)$ -QBSC $_{I_{acc}}$ α protocols. The protocol is as follows: Alice chooses a set of $O(n^4)$ unitaries independently according to the Haar measure (approximated) and announces the resulting set \mathcal{U} to Bob. They then perform LOCKCOM(n, \mathcal{U}). Our analysis shows that this variant is secure against Bob with high probability. Unfortunately, the protocol is inefficient both in terms of computation and communication. It remains open to find an efficient constructive scheme with those parameters.

In contrast, for only two bases, an efficient construction exists and uses the identity and the Hadamard transform as unitaries. From [18] (see also [21]) it then follows that LOCKCOM($n, \{\mathbb{I}^{\otimes n}, H^{\otimes n}\}$) is an $(n, 1, n/2)$ -QBSC $_{I_{acc}}$ protocol. As shown in [22], this protocol can be made *cheat sensitive* [23] for Bob, i.e., any nonzero information gain by Bob will be detected by Alice with nonzero probability.

A drawback of weakening the security requirement is that LOCKCOM protocols are not necessarily composable. Thus, if LOCKCOM is used as a subprotocol in a larger protocol, the security of the resulting scheme has to be evaluated on a case by case basis. However, LOCKCOM protocols are secure when executed in parallel. This is a consequence of the definition of Alice's security parameter and the additivity of the accessible information [24], and sufficient for many cryptographic purposes.

Conclusion.—We have introduced a framework for quantum commitments to a string of bits and shown that under strong security requirements, nontrivial protocols do not exist. A property of quantum states known as *locking*, however, allowed us to propose meaningful protocols for a weaker security demand.

We thank J. Barrett, A. Broadbent, I. Damgård, A. Kent, S. Massar, R. Renner, R. Spekkens, and R. de Wolf for discussions. We also thank R. Jain for discussion on his work [25], where, following Ref. [17], he used a different method to prove that (n, a, b) -QBSC $_{\chi, s}$ with $a + 16b + 31 < n$, do not exist. We also thank Germany's DAAD, the UK's EPSRC, Magdalene College Cambridge, CFI, CIAR, CIPI, CRC, NSERC, PREA and OIT, the NWO vici project 2004–2009, and the EU Projects No. RESQ IST-2001-37559, QAP IST 015848, and the FP6-FET Integrated Project SCALA, CT-015714.

*Electronic address: matthias.christandl@qubit.org

†Electronic address: wehner@cw.nl

[1] C. H. Bennett, G. Brassard, C. Crépeau, and M.-H. Skubiszewska, in *Proceedings of CRYPTO '91* (Springer, New York, 1991), pp. 351–366.

- [2] C. Crépeau, *J. Mod. Opt.* **41**, 2455 (1994).
- [3] A. C.-C. Yao, in *Proceedings of 20th ACM STOC* (ACM, New York, 1995), pp. 67–75.
- [4] C. Crépeau, J. van de Graaf, and A. Tapp, in *Proceedings of CRYPTO '95* (Springer, New York, 1995), pp. 110–123; J. Kilian, in *Proceedings of the 20th ACM STOC* (ACM, New York, 1988), pp. 20–31.
- [5] O. Goldreich, *Basic Tools* (Cambridge University Press, Cambridge, England, 2001).
- [6] M. Blum, *SIGACT News* **15**, 23 (1983).
- [7] C. H. Bennett and G. Brassard, in *Proceedings of the IEEE ICCSS* (IEEE, New York, 1984), pp. 175–179; G. Brassard and C. Crépeau, in *Proceedings of CRYPTO '90* (Springer, New York, 1990), pp. 49–61; G. Brassard, C. Crépeau, R. Jozsa, and D. Langlois, in *Proceedings of the 34th IEEE FOCS* (IEEE, New York, 1993), pp. 362–371.
- [8] H.-K. Lo and H. F. Chau, *Phys. Rev. Lett.* **78**, 3410 (1997); *Physica (Amsterdam)* **120D**, 177 (1998); D. Mayers, *Phys. Rev. Lett.* **78**, 3414 (1997).
- [9] D. Mayers, quant-ph/9603015.
- [10] A. Kitaev, D. Mayers, and J. Preskill, *Phys. Rev. A* **69**, 052326 (2004).
- [11] R. Spekkens and T. Rudolph, *Phys. Rev. A* **65**, 012310 (2001).
- [12] A. Kent, *Phys. Rev. Lett.* **90**, 237901 (2003).
- [13] R. Renner, Ph.D. thesis, ETH Zürich [quant-ph/0512258].
- [14] H. Barnum and E. Knill, *J. Math. Phys. Sci.* **43**, 2097 (2002).
- [15] R. König, U. Maurer, and R. Renner, *IEEE Trans. Inf. Theory* **51**, 2391 (2005); R. Renner and R. König, in *Proceedings of TCC '05*, edited by J. Kilian, Lecture Notes in Computer Science Vol. 3378 (Springer-Verlag, Berlin, 2005), pp. 407–425.
- [16] Alice learns x , but cannot pick it: she committed to a superposition and x is chosen randomly by measurement.
- [17] H. Buhrman, M. Christandl, P. Hayden, H.-K. Lo, and S. Wehner, quant-ph/0504078.
- [18] D. DiVincenzo, M. Horodecki, D. Leung, J. Smolin, and B. Terhal, *Phys. Rev. Lett.* **92**, 067902 (2004).
- [19] This can be seen as follows. Let \tilde{p}_x denote the probability that Alice reveals x successfully. Then, $\tilde{p}_x \leq \sum_r \tilde{p}_{x,r}$, where $\tilde{p}_{x,r}$ is the probability that x is accepted by Bob when the reveal information was r . Let ρ denote the state of Bob's system. Summation over x yields $\sum_x \tilde{p}_x \leq \sum_{x,r} \tilde{p}_{x,r} = \sum_{x,r} \text{Tr}|x\rangle\langle x|U_r^\dagger \rho U_r = \sum_r \text{Tr} \rho = 2^a$.
- [20] P. Hayden, D. Leung, P. Shor, and A. Winter, *Commun. Math. Phys.* **250**, 371 (2004).
- [21] M. Christandl and A. Winter, *IEEE Trans. Inf. Theory* **51**, 3159 (2005).
- [22] M. Christandl, Ph.D. thesis, University of Cambridge [quant-ph/0604183].
- [23] D. Aharonov, A. Ta-Shma, U. Vazirani, and A. Yao, in *Proceedings of the 32nd ACM STOC* (ACM, New York, 2000), pp. 705–714; L. Hardy and A. Kent, *Phys. Rev. Lett.* **92**, 157901 (2004).
- [24] D. P. DiVincenzo, D. W. Leung, and B. M. Terhal, *IEEE Trans. Inf. Theory* **48**, 580 (2002); A. S. Holevo, *Probl. Inf. Transm.* **9**, 110 (1973).
- [25] R. Jain, quant-ph/0506001.