

Finite Blocklength Converse Bounds for Quantum Channels

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Abstract—We derive upper bounds on the rate of transmission of classical information over quantum channels by block codes with a given blocklength and error probability, for both entanglement-assisted and unassisted codes, in terms of a unifying framework of quantum hypothesis testing with restricted measurements. Our bounds do not depend on any special property of the channel (such as memorylessness) and generalize both a classical converse of Polyanskiy, Poor, and Verdú as well as a quantum converse of Renner and Wang, and have a number of desirable properties. In particular, our bound on entanglement-assisted codes is a semidefinite program and for memoryless channels, its large blocklength limit is the well-known formula for entanglement-assisted capacity due to Bennett, Shor, Smolin, and Thapliyal.

Index Terms—Channel coding, quantum channels, block codes, finite blocklength, quantum entanglement.

I. INTRODUCTION

THIS work is concerned with the transmission of classical information over quantum channels by means of block codes. This is a central subject of study in quantum information theory, and the *asymptotic* rates of transmission in the large blocklength limit, for various types of code and channel, are the subject of celebrated theorems and intriguing open problems. A more fundamental problem, of both theoretical and practical interest, is to obtain upper (or *converse*) and lower (or *achievability*) bounds on the optimal transmission rate for a given error probability ϵ and *finite* blocklength n .

Without assumptions on the structure of the operation implemented by n channel uses (e.g. independence), there is only a notational difference between coding for n uses of a channel and coding for one use of a larger composite channel, so bounds which apply in this setting are also known as

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‘one-shot’ bounds. These are the subject of a number of recent results in quantum information [1]–[5] and remain an active topic of research in classical information [6], [7]. All bounds referred to in the remainder of this introduction are of this type.

Mosonyi and Datta [1], Wang and Renner [2] and Renes and Renner [3] have given converse and achievability bounds for classical-quantum channels. In [4] Datta and Hsieh derive converse and achievability results for *entanglement-assisted coding* over quantum channels in terms of smoothed min- and max-entropies.

Polyanskiy, Poor and Verdú [6] identified a very general approach to *classical* converse bounds which they call the ‘meta-converse’. The bounds they obtain are given in terms of a classical hypothesis testing problem. In this paper we further generalise this approach to the quantum setting, obtaining novel converse bounds for both entanglement-assisted and unassisted coding over general quantum channels. Our bounds are given in terms of a *quantum* hypothesis testing problem on a bipartite system. In the bounds for unassisted codes the measurements used for the hypothesis test obey certain locality restrictions with respect to the bipartition.

Section II gives a brief review of the mathematical framework that we work with, introduces the classes of restricted operations we use (II-A) and quantum hypothesis testing with these restrictions (II-B), and defines precisely entanglement-assisted and unassisted codes (II-C). Section III states our converse bounds and summarises the properties of these bounds, which we establish in Section V. Section IV proves the converse bounds via a quantum ‘meta-converse’ result.

As an example of the application of our bound for entanglement-assisted codes, in Section VI we show how to compute it exactly for n uses of a depolarising channel. In Section VII we discuss the relationship of this work to existing results on strong converse bounds for quantum channels, and to security proofs in the noisy-storage model. We conclude in Section VIII, where we mention some open problems.

II. PRELIMINARIES

As usual, a quantum system Q is associated with a Hilbert space \mathcal{H}_Q . By the dimension of the system $\dim(Q)$ we mean the dimension of the associated space. This work deals only with finite-dimensional systems. The space of linear maps from \mathcal{H}_Q to \mathcal{H}_R , we denote by $\mathcal{L}(\mathcal{H}_Q, \mathcal{H}_R)$, with the abbreviation $\mathcal{L}(\mathcal{H}_Q)$ for the space $\mathcal{L}(\mathcal{H}_Q, \mathcal{H}_Q)$ of linear operators on \mathcal{H}_Q .

By a *state* of Q we mean a density (i.e. positive semidefinite, trace one) operator in $\mathcal{L}(\mathcal{H}_Q)$. We denote the set of all states of Q by $\mathbf{states}(Q)$. We omit tensor products, when the subscripts make it clear on which systems the operators act e.g. $\rho_A \sigma_B = \rho_A \otimes \sigma_B$.

By a *sub-operation* with input system A and output system B we mean a linear map from $\mathcal{L}(\mathcal{H}_A)$ to $\mathcal{L}(\mathcal{H}_B)$ which is completely positive and trace non-increasing (when acting on positive operators). We denote the set of these by $\mathbf{subops}(A \rightarrow B)$. If a sub-operation is trace preserving, then it is called an *operation*. We denote the subset of operations in $\mathbf{subops}(A \rightarrow B)$ by $\mathbf{ops}(A \rightarrow B)$.

As a completely positive map, any sub-operation $\mathcal{N}_{B|A} \in \mathbf{subops}(A \rightarrow B)$ has a (non-unique) representation

$$\mathcal{N}_{B|A} : X_A \mapsto \sum_i M_i X_A M_i^\dagger \quad (1)$$

in terms of linear operators $M_i \in \mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$ called *Kraus operators*. The trace non-increasing condition is equivalent to $\sum_i M_i^\dagger M_i \leq \mathbb{1}_A$, and $\mathcal{N}_{B|A}$ is trace preserving iff $\sum_i M_i^\dagger M_i = \mathbb{1}_A$.

Given a linear map $\mathcal{N}_{B|A} : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_B)$ its *adjoint* map is defined to be the unique linear map $\mathcal{N}_{B|A}^\dagger : \mathcal{L}(\mathcal{H}_B) \rightarrow \mathcal{L}(\mathcal{H}_A)$ such that, for all $X_A \in \mathcal{L}(\mathcal{H}_A)$ and $Y_B \in \mathcal{L}(\mathcal{H}_B)$, we have $\text{Tr}_B Y_B^\dagger \mathcal{N}_{B|A} X_A = \text{Tr}_A (\mathcal{N}_{B|A}^\dagger Y_B)^\dagger X_A$. If $\mathcal{N}_{B|A}$ is an operation, then $\mathcal{N}_{A|B}^\dagger$ is completely positive and unital (i.e. it maps the identity operator to the identity operator).

Processes which produce a classical outcome in some set W , can be represented by an *instrument*. For our purposes, it suffices to consider finite W and we can represent an instrument by a collection of suboperations indexed by W , $\{\mathcal{N}_{B|A}^{(w)}\}_{w \in W}$, which sum to an operation: $\sum_{w \in W} \mathcal{N}_{B|A}^{(w)} \in \mathbf{ops}(A \rightarrow B)$. If the instrument is applied to a state ρ_A then the probability of outcome w is $\text{Tr}_B \mathcal{N}_{B|A}^{(w)} \rho_A$, and the state of B conditioned on outcome w is $(\mathcal{N}_{B|A}^{(w)} \rho_A) / (\text{Tr}_B \mathcal{N}_{B|A}^{(w)} \rho_A)$. If $M_i^{(w)}$ are Kraus operators for $\mathcal{N}_{B|A}^{(w)}$, then $\text{Tr}_B \mathcal{N}_{B|A}^{(w)} \rho_A = \sum_i \text{Tr}_B M_i^{(w)} \rho_A (M_i^{(w)})^\dagger = \text{Tr}_A E(w)_A \rho_A$ where $E(w)_A := \sum_i (M_i^{(w)})^\dagger M_i^{(w)}$, and the equation follows from the linearity and cyclic property of trace. The *POVM* (positive operator-valued measure) $\{E(w)_A\}_{w \in W}$ and the state ρ_A determine the distribution of the outcome. The fact that $\{\mathcal{N}_{B|A}^{(w)}\}_{w \in W}$ is an instrument implies that POVM elements satisfy $E(w)_A \geq 0$ and $\sum_{w \in W} E(w)_A = \mathbb{1}_A$.

It is convenient to assume that every quantum system Q comes equipped with a canonical orthonormal basis which we call the *classical basis*, and whose members we denote by $|i\rangle_Q$ for $i = 1, \dots, \dim(Q)$. Given two systems Q and \tilde{Q} of the same dimension, we denote by $\text{id}_{\tilde{Q}|Q}$ the “identity” operation from Q to \tilde{Q} which is defined via its action on the classical bases of the systems, thus

$$\text{id}_{\tilde{Q}|Q} : |i\rangle\langle j|_Q \mapsto |i\rangle\langle j|_{\tilde{Q}}. \quad (2)$$

We also define the *transpose map* on a system Q by its action on the classical basis

$$\text{t}_{Q|Q} : |i\rangle\langle j|_Q \mapsto |j\rangle\langle i|_Q. \quad (3)$$

It is important to note that $\text{t}_{Q|Q}$ is positive but not completely positive, so it is not an operation. Given an operator X_Q we will also make use of the standard notation for its transpose $X_Q^T := \text{t}_{Q|Q} X_Q$. Similarly, the complex conjugate X_Q^* of X_Q is defined by taking the classical basis to be real. Therefore, the adjoint X_Q^\dagger of an operator X_Q satisfies $X_Q^\dagger = (X_Q^T)^*$.

A. Classes of Operation on Bipartite Systems

Let $\mathbf{ops}^\Omega(A : B \rightarrow A' : B')$ denote the set of all operations which take states of the bipartite system $A : B$ to states of $A' : B'$, and which belong to class Ω . We insert the colon to make explicit the relevant bipartitions of the input and output systems.

We call an operation a *measurement operation* if, for any input, its output is diagonal in the classical basis of its output system(s). It is worth emphasising that the classical basis of a composite system is the *product basis* formed from the classical bases of its constituents. We denote the subset of measuring operations in $\mathbf{ops}^\Omega(A : B \rightarrow A' : B')$ by $\mathbf{mops}^\Omega(A : B \rightarrow A' : B')$. Measuring operations are also called “quantum-classical,” or “q-c,” operations in the literature.

An operation $\mathcal{N}_{A'B'|AB} \in \mathbf{ops}(A : B \rightarrow A' : B')$ belongs to the class **PPT** if it is positive-partial-transpose preserving, i.e. if $\text{t}_{B'|B} \mathcal{N}_{A'B'|AB} \text{t}_{B|B}$ is completely positive.

$\mathcal{N}_{A'B'|AB}$ belongs to **LC1** if it can be implemented by local operations and one-way classical communication from Alice to Bob: Alice performs any instrument on her side, generating a classical outcome a which she sends to Bob. Bob uses a to determine which operation he applies. Such an operation can be written in the form

$$\mathcal{N}_{A'B'|AB} = \sum_a \mathcal{F}_{A'|A}^{(a)} \mathcal{D}_{B'|B}^{(a)} \quad (4)$$

where $\sum_a \mathcal{F}_{A'|A}^{(a)} \in \mathbf{ops}(A \rightarrow A')$ and where, for each a , $\mathcal{F}_{A'|A}^{(a)} \in \mathbf{subops}(A \rightarrow A')$ and $\mathcal{D}_{B'|B}^{(a)} \in \mathbf{ops}(B \rightarrow B')$. Throughout, we will omit tensor products between operations when it is clear which systems they act on from the subscripts.

$\mathcal{N}_{A'B'|AB}$ belongs to **L** if it can be implemented by local operations and shared randomness, which means it can be written

$$\mathcal{N}_{A'B'|AB} = \sum_r p_r \mathcal{F}_{A'|A}^{(r)} \mathcal{D}_{B'|B}^{(r)} \quad (5)$$

where, for each r , $\mathcal{F}_{A'|A}^{(r)} \in \mathbf{ops}(A \rightarrow A')$ and $\mathcal{D}_{B'|B}^{(r)} \in \mathbf{ops}(B \rightarrow B')$ and p_r is the probability of the shared random variable being equal to r .

These classes of operations are all closed under composition. Furthermore, for any measurement operation $\mathcal{M}_{A''B''|A'B'} \in \mathbf{mops}^\Omega(A' : B' \rightarrow A'' : B'')$ and operation $\mathcal{W}_{A'B'|AB} \in \mathbf{ops}^\Omega(A : B \rightarrow A' : B')$ (where $\Omega \in \{\mathbf{L}, \mathbf{LC1}, \mathbf{PPT}, \mathbf{ALL}\}$) we have the closure property

$$\mathcal{M}_{A''B''|A'B'} \mathcal{W}_{A'B'|AB} \in \mathbf{mops}^\Omega(A : B \rightarrow A'' : B''). \quad (6)$$

Note, however, that following a measurement operation with an operation which is not measuring won't necessarily result in a measurement operation. All the classes mentioned are

closed under convex combination. Furthermore, they form a hierarchy $\mathbf{ALL} \supset \mathbf{PPT} \supset \mathbf{LC1} \supset \mathbf{L}$ [8], [9].

B. Quantum Hypothesis Testing With Restricted Measurements

In a classical hypothesis testing problem (with simple hypotheses and finite sample space) there are two hypotheses H_i , $i \in \{0, 1\}$, of the form ‘the random variable R has distribution $P^{(i)}$ ’. A statistical test T can be specified by the giving the probabilities $T(r) = \Pr(\text{accept } H_0 | T, R = r)$.

The ‘type-I error’ of T is

$$\alpha(P^{(0)}, T) = \Pr(\text{accept } H_1 | H_0, T) \quad (7)$$

$$= 1 - \sum_r P^{(0)}(r)T(r) \quad (8)$$

while the ‘type-II error’ of T is

$$\beta(P^{(1)}, T) = \Pr(\text{accept } H_0 | H_1, T) \quad (9)$$

$$= \sum_r P^{(1)}(r)T(r). \quad (10)$$

Definition 1:

$$\beta_\epsilon(P^{(0)} \| P^{(1)}) := \min \beta(P^{(1)}, T) \quad (11)$$

subject to

$$\alpha(P^{(0)}, T) \leq \epsilon, \quad (12)$$

$$\forall r : 0 \leq T(r) \leq 1. \quad (13)$$

In a quantum hypothesis testing problem (with simple hypotheses) there are two hypotheses H_i , $i \in \{0, 1\}$, of the form ‘the state of system Q is $\tau_Q^{(i)}$ ’. In order to distinguish between these situations it is necessary to perform a measurement on Q , the outcome of which is then subjected to a classical hypothesis test.

If the measurement operation is $\mathcal{M}_{C|Q}$ and the classical test has probability $T(r)$ of accepting when the outcome is r , then the overall probability of acceptance when the state of Q is τ_Q is

$$\sum_r T(r) \text{Tr}_C |r\rangle\langle r|_C \mathcal{M}_{C|Q} \tau_Q = \text{Tr}_Q T_Q \tau_Q \quad (14)$$

where $T_Q = \mathcal{M}_{Q|C}^\dagger (\sum_r T(r) |r\rangle\langle r|_C)$ is the POVM element corresponding to acceptance of H_0 . To see that it is a POVM element, we can use the fact that $0 \leq \sum_r T(r) |r\rangle\langle r|_C \leq 1$, and that the adjoint map $\mathcal{M}_{Q|C}^\dagger$ of an operation is completely positive and unital.

With no restriction on the measurement operation (i.e. for class \mathbf{ALL}), T_Q can be any valid POVM element. Therefore, we are justified in defining

Definition 2:

$$\beta_\epsilon^{\mathbf{ALL}}(\tau_Q^{(0)} \| \tau_Q^{(1)}) := \min \text{Tr}_Q \tau_Q^{(1)} T_Q \quad (15)$$

subject to

$$1 - \text{Tr}_Q \tau_Q^{(0)} T_Q \leq \epsilon, \quad (16)$$

$$\forall r : 0 \leq T_Q \leq 1. \quad (17)$$

Suppose $\tau_Q^{(0)}$ and $\tau_Q^{(1)}$ commute, and $|\eta_i\rangle\langle\eta_i|_Q$ is a common eigenbasis for them. Defining the operation

$$\mathcal{K}_{Q|Q} : X_Q \mapsto \sum_{i=1}^{\dim(Q)} |\eta_i\rangle\langle\eta_i|_Q X_Q |\eta_i\rangle\langle\eta_i|_Q \quad (18)$$

which completely dephases the system in this eigenbasis, we have $\mathcal{K}_{Q|Q} \tau_Q^{(0)} = \tau_Q^{(0)}$ and $\mathcal{K}_{Q|Q} \tau_Q^{(1)} = \tau_Q^{(1)}$. Since $\mathcal{K}_{Q|Q}$ is its own adjoint map, this means that there a test T_Q , optimal for the hypothesis test, which itself satisfies $\mathcal{K}_{Q|Q} T_Q = T_Q$. That is, it too is diagonal in the common eigenbasis of the two states. In particular, if both $\tau_Q^{(0)}$ and $\tau_Q^{(1)}$ are diagonal in the classical basis, then we can assume without loss of generality that the test is too, whereupon everything reduces to the classical case.

We extend this definition to restricted measurements on bipartite systems by

Definition 3: For $\Omega \neq \mathbf{ALL}$,

$$\beta_\epsilon^\Omega(\tau_{AB}^{(0)} \| \tau_{AB}^{(1)}) := \inf \beta_\epsilon(\mathcal{M}_{A'B'|AB} \tau_{AB}^{(0)} \| \mathcal{M}_{A'B'|AB} \tau_{AB}^{(1)}) \quad (19)$$

subject to

$$\mathcal{M}_{A'B'|AB} \in \mathbf{mops}^\Omega(A : B \rightarrow A' : B'), \quad (20)$$

$$\text{for arbitrary } A', B'. \quad (21)$$

That is, we reduce the quantum to the classical case by optimising over all measurement operations belonging to the allowed class Ω . Based on this quantity, we define the Ω -hypothesis-testing relative entropy

$$D_\epsilon^\Omega(\tau_{AB}^{(0)} \| \tau_{AB}^{(1)}) := -\log \beta_\epsilon^\Omega(\tau_{AB}^{(0)} \| \tau_{AB}^{(1)}). \quad (22)$$

A trivial but very useful result is the following.

Proposition 4 (Data Processing Inequality): If Ω is closed under composition, then for any operation $\mathcal{N}_{A'B'|AB} \in \mathbf{ops}^\Omega(A : B \rightarrow A' : B')$

$$D_\epsilon^\Omega(\mathcal{N}_{A'B'|AB} \tau_{AB}^{(0)} \| \mathcal{N}_{A'B'|AB} \tau_{AB}^{(1)}) \leq D_\epsilon^\Omega(\tau_{AB}^{(0)} \| \tau_{AB}^{(1)}). \quad (23)$$

Corollary 5: If there is also an operation $\mathcal{N}'_{AB|A'B'} \in \mathbf{ops}^\Omega(A' : B' \rightarrow A : B)$ such that $\mathcal{N}'_{AB|A'B'} \mathcal{N}_{A'B'|AB} \tau_{AB}^{(0)} = \tau_{AB}^{(0)}$ and $\mathcal{N}'_{AB|A'B'} \mathcal{N}_{A'B'|AB} \tau_{AB}^{(1)} = \tau_{AB}^{(1)}$ then

$$D_\epsilon^\Omega(\mathcal{N}_{A'B'|AB} \tau_{AB}^{(0)} \| \mathcal{N}_{A'B'|AB} \tau_{AB}^{(1)}) = D_\epsilon^\Omega(\tau_{AB}^{(0)} \| \tau_{AB}^{(1)}). \quad (24)$$

Suppose that the optimal (classical) hypothesis test which acts on the measurement result in Definition 3 has probability $T(a, b)$ of accepting H_0 when the measurement operation produces the outcome $|a\rangle\langle a|_{A'} |b\rangle\langle b|_{B'}$. Then the overall probability of the test accepting H_0 when the state of AB is τ_{AB} is

$$\sum_{a,b} T(a, b) \text{Tr}_{A'B'} |a\rangle\langle a|_{A'} |b\rangle\langle b|_{B'} \mathcal{M}_{A'B'|AB} \tau_{AB} \quad (25)$$

$$= \text{Tr}_{AB} T_{AB} \tau_{AB} \quad (26)$$

where (with $\mathcal{M}'_{AB|A'B'}$ the adjoint map for $\mathcal{M}_{A'B'|AB}$),

$$T_{AB} = \mathcal{M}'_{AB|A'B'} \left(\sum_{a,b} T(a, b) |a\rangle\langle a|_{A'} |b\rangle\langle b|_{B'} \right). \quad (27)$$

T_{AB} is the POVM element (which we call simply a ‘test’) corresponding to acceptance of hypothesis H_0 in a quantum hypothesis test which can be implemented by a measurement operation in Ω followed by a classical hypothesis test in the joint outcome. We denote the set of such tests on AB by $\mathbf{T}^\Omega(A : B)$.

As shown in [10], $\mathbf{T}^{\text{PPT}}(A : B)$ consists of all POVM elements T_{AB} (that is, $0 \leq T_{AB} \leq \mathbb{1}_{AB}$), satisfying

$$0 \leq \text{tr}_B T_{AB} \leq \mathbb{1}_A. \quad (28)$$

Working through the definitions, we see that $\mathbf{T}^{\text{L}}(A : B)$ is the convex hull of all POVM elements of the form

$$\sum_{a,b} T(a,b) E(a)_A D(b)_B \quad (29)$$

where $\{E(a)_A\}$ is a POVM on A, $\{D(b)_B\}$ is a POVM on B, and $0 \leq T(a,b) \leq 1$. $\mathbf{T}^{\text{LC1}}(A : B)$ is the convex hull of all POVM elements of the form

$$\sum_{a,b} T(a,b) E(a)_A D^a(b)_B \quad (30)$$

where $\{E(a)_A\}$ is a POVM on A, and for each a , $\{D^a(b)_B\}$ is a POVM on B, and $0 \leq T(a,b) \leq 1$. (We can use the results of [11] to show that, given our assumption that A and B are finite dimensional, it suffices to take local POVMs with a finite number of outcomes in these last two statements.)

Proposition 6:

$$\beta_\epsilon^\Omega(\tau_{AB}^{(0)} \| \tau_{AB}^{(1)}) = \inf \text{Tr} \tau_{AB}^{(1)} T_{AB} \quad (31)$$

subject to

$$\text{Tr}_{AB} \tau_{AB}^{(0)} T_{AB} \geq 1 - \epsilon, \quad (32)$$

$$T_{AB} \in \mathbf{T}^\Omega(A : B). \quad (33)$$

Remark 7: For the classes of operations considered in this paper (**ALL**, **PPT**, **LC1** and **L**), $\mathbf{T}^\Omega(A : B)$ is a closed set, and so for these classes the infimum in Proposition 6 and Definition 3 can be replaced by a minimum.

C. Codes

As usual, a use (or uses) of a quantum channel with input system A and output system B is represented by an operation $\mathcal{E}_{B|A} \in \mathbf{ops}(A \rightarrow B)$.

Definition 8: In an **entanglement-assisted code** of size M , the sender and receiver have systems A_E and B_E in an entangled state $\Psi_{A_E B_E}$, and for each message $w \in \{1, 2, \dots, M\}$ there is an encoding operation $\mathcal{C}(w)_{A|A_E} \in \mathbf{ops}(A_E \rightarrow A)$. Following the use(s) of the channel, the decoder performs a POVM $D_{B_E B}$ on $B_E B$ to obtain the decoded message. $D(\hat{w})_{B_E B}$ is the POVM element corresponding to the decoded message being \hat{w} .

Definition 9: An **unassisted code** can be viewed as a degenerate case of an entanglement-assisted code where the decoding measurement operates only on the channel output B. Since B_E is completely ignored, there is no loss of generality if we take B_E and A_E to be trivial, one-dimensional systems, so that $\mathcal{C}(w)_{A|A_E}$ is completely specified by its output $\rho(w)_A$ on A.

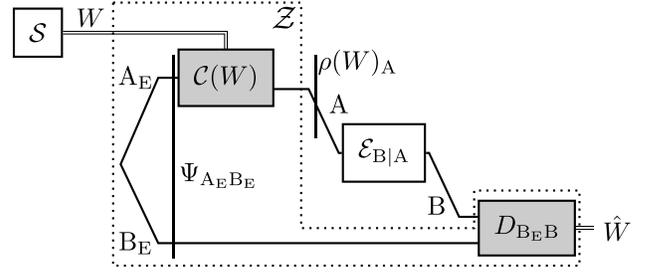


Fig. 1. An entanglement-assisted code \mathcal{Z} transmitting a message W produced by a source S , via a channel use \mathcal{E} . The average channel input induced by the source and encoding is $\rho_A = \sum_{w=1}^M \mathcal{S}(w) \rho(w)_A$.

Figure 1 illustrates an entanglement-assisted code \mathcal{Z} transmitting a message W produced by a source S via a channel use $\mathcal{E}_{B|A}$. The message W and the outcome \hat{W} of the decoding POVM are classical random variables. The source is specified by the probabilities

$$\mathcal{S}(w) := \Pr(W = w|S). \quad (34)$$

The probability of error (which depends on the source, code and channel) is

$$\Pr(\hat{W} \neq W | \mathcal{E}, \mathcal{Z}, \mathcal{S}). \quad (35)$$

For $M \in \mathbb{N}$ let \mathcal{S}_M denote a source with M equiprobable messages i.e. $\mathcal{S}_M(w) = 1/M$.

Definition 10: We call a size M code \mathcal{Z} an (M, ϵ, ρ_A) code for \mathcal{E} , if its average error probability for M equiprobable messages satisfies $\Pr(\hat{W} \neq W | \mathcal{E}, \mathcal{Z}, \mathcal{S}_M) \leq \epsilon$, and the average channel input state induced by using the code with equiprobable messages is ρ_A .

We denote by $M_\epsilon(\mathcal{E}_{B|A}, \rho_A)$ the largest M such that there is an (M, ϵ, ρ_A) unassisted code, and by $M_\epsilon(\mathcal{E}_{B|A})$ the largest M such that there exists an ρ_A such that there is an (M, ϵ, ρ_A) unassisted code.

$M_\epsilon^E(\mathcal{E}_{B|A}, \rho_A)$ and $M_\epsilon^E(\mathcal{E}_{B|A})$ denote the corresponding quantities for entanglement-assisted codes.

Remark 11: Clearly,

$$M_\epsilon^E(\mathcal{E}_{B|A}) = \max_{\rho_A \in \mathbf{states}(A)} M_\epsilon^E(\mathcal{E}_{B|A}, \rho_A) \quad (36)$$

and

$$M_\epsilon(\mathcal{E}_{B|A}) = \max_{\rho_A \in \mathbf{states}(A)} M_\epsilon(\mathcal{E}_{B|A}, \rho_A) \quad (37)$$

III. SUMMARY OF RESULTS

In [6] Polyanskiy, Poor and Verdú showed that many existing *classical* converse results can be easily derived from a finite blocklength converse ([6, Th. 27]) which we will call the *PPV converse*. It is obtained by a simple and conceptually appealing argument relating coding to hypothesis testing on the joint distribution of the channel input and channel output. Our bounds are given in terms of a quantum hypothesis testing problem on a bipartite system, consisting of B (the output system for the channel operation) and \hat{A} (a copy of the input system). To compactly describe the hypotheses, we first introduce a little notation.

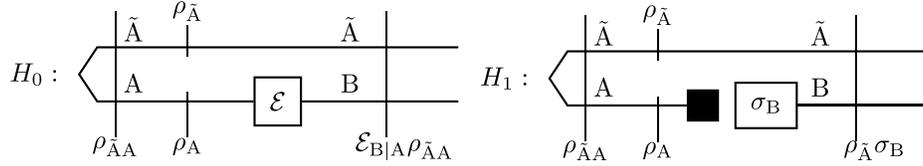


Fig. 2. The quantum hypothesis testing problem which appears in our bounds.

Definition 12: For any two systems Q, \tilde{Q} of the same dimension, let $\Phi_{\tilde{Q}Q} := \sum_{i,j=1}^{\dim(Q)} |i\rangle_{\tilde{Q}} |i\rangle_Q \langle j|_{\tilde{Q}} \langle j|_Q$.

Remark 13: $\Phi_{\tilde{Q}Q}$ is $\dim(Q)$ times the isotropic, maximally entangled state of $\tilde{Q}Q$. Note that $\text{Tr}_{\tilde{Q}} \Phi_{\tilde{Q}Q} = \mathbb{1}_Q$ and $\text{Tr}_Q \Phi_{\tilde{Q}Q} = \mathbb{1}_{\tilde{Q}}$. We will make use, more than once, of the basic fact that for any linear operator M_Q on \mathcal{H}_Q , $M_Q \Phi_{\tilde{Q}Q} = M_Q^T \Phi_{\tilde{Q}Q}$, where $M_Q := \text{id}_{\tilde{Q}} M_Q$.

Definition 14: Given a state ρ_A , let \tilde{A} be a copy of system A , and define a canonical purification of ρ_A on $\tilde{A}A$ by

$$\rho_{\tilde{A}A} := \rho_A^{\frac{1}{2}} \Phi_{\tilde{A}A} \rho_A^{\frac{1}{2}}. \quad (38)$$

Remark 15: By Remark 13, we have $\text{Tr}_{\tilde{A}} \rho_{\tilde{A}A} = \rho_A$, and we find that the marginal state of \tilde{A} is

$$\rho_{\tilde{A}} := \text{Tr}_A \rho_{\tilde{A}A} = \text{id}_{\tilde{A}|A} \rho_A^T. \quad (39)$$

Throughout the paper, we regard $\rho_{\tilde{A}A}$ and $\rho_{\tilde{A}}$ as functions of ρ_A .

As shown in Fig. 2, the hypotheses specify quantum states of a bipartite system $\tilde{A}B$, where B is the output system of $\mathcal{E}_{B|A}$ and \tilde{A} is isomorphic to its input system. Hypothesis H_0 is that $\tilde{A}B$ is in the state $\mathcal{E}_{B|A} \rho_{\tilde{A}A}$, whereas hypothesis H_1 is that $\tilde{A}B$ is in the product state $\rho_{\tilde{A}} \sigma_B$.

Our main converse bounds are quantum generalisations of [6, Th. 27], for entanglement-assisted, and for unassisted, codes. To state them, let us introduce

Definition 16:

$$\beta_\epsilon^\Omega(\mathcal{E}_{B|A}, \rho_A) := \max_{\sigma_B \in \text{states}(B)} \beta_\epsilon^\Omega(\mathcal{E}_{B|A} \rho_{\tilde{A}A} \| \rho_{\tilde{A}} \sigma_B), \quad (40)$$

which is given in terms of Definition 3. In words, this is the minimum type-II error of all tests in class Ω which have type-I error no greater than ϵ for the hypothesis testing problem depicted in Figure 2, maximised over all σ_B .

The relationship between the mutual information and standard relative entropy, informs

Definition 17:

$$I_\epsilon^\Omega(\mathcal{E}_{B|A}, \rho_A) := \min_{\sigma_B \in \text{states}(B)} D_\epsilon^\Omega(\mathcal{E}_{B|A} \rho_{\tilde{A}A} \| \rho_{\tilde{A}} \sigma_B) \quad (41)$$

$$= -\log \beta_\epsilon^\Omega(\mathcal{E}_{B|A}, \rho_A). \quad (42)$$

We note again that $\rho_{\tilde{A}A}$ is here the canonical pure state of Definition 14 and is to be regarded as a function of ρ_A .

With these and Definition 10, we can now state the main results of this paper in a compact form:

Theorem 18 (Entanglement-Assisted Converse):

$$\log M_\epsilon^E(\mathcal{E}_{B|A}, \rho_A) \leq I_\epsilon^{\text{ALL}}(\mathcal{E}_{B|A}, \rho_A), \text{ and so} \quad (43)$$

$$\log M_\epsilon^E(\mathcal{E}_{B|A}) \leq \max_{\rho_A \in \text{states}(A)} I_\epsilon^{\text{ALL}}(\mathcal{E}_{B|A}, \rho_A). \quad (44)$$

Theorem 19 (Unassisted Converse): For any class of operations Ω containing \mathbf{L} , we have

$$\log M_\epsilon(\mathcal{E}_{B|A}, \rho_A) \leq I_\epsilon^\Omega(\mathcal{E}_{B|A}, \rho_A), \text{ and so} \quad (45)$$

$$\log M_\epsilon(\mathcal{E}_{B|A}) \leq \max_{\rho_A \in \text{states}(A)} I_\epsilon^\Omega(\mathcal{E}_{B|A}, \rho_A). \quad (46)$$

The proof of both of these is given in the next section (IV). In section V we prove a number of properties of the bounds, which we first summarise here. The converse for entanglement-assisted codes in Theorem 18 has a number of desirable properties:

- 1) Like the bound of Datta and Hsieh [4], it is *asymptotically tight* for memoryless channels. That is to say, analysing the large block length behaviour of the bound for memoryless channels recovers the converse part of the single-letter formula for entanglement-assisted capacity proven by Bennett, Shor, Smolin and Thapliyal [12]. This is shown in subsection V-D.
- 2) For a fixed blocklength, the converse of Datta and Hsieh [4] grows slowly, but without bound, as $\epsilon \rightarrow 0$ whereas our converse is a decreasing function of ϵ .
- 3) Generalising results of Polyanskiy [7] we show that $\beta_\epsilon^{\text{ALL}}(\mathcal{E}_{B|A} \rho_{\tilde{A}A} \| \rho_{\tilde{A}} \sigma_B)$ is convex in ρ_A and concave in σ_B . This enables one to use symmetries of the channel to restrict the optimisation over ρ_A and σ_B to states with corresponding symmetries, as we show in subsection V-E.
- 4) In subsection V-A, we give an explicit formulation of the bound as *semidefinite program* (SDP) which is a natural generalisation of the linear program (LP) given in [13] for the PPV converse.

Regarding our converse for unassisted codes (Theorem 19): In subsection V-C the Wang-Renner bound is shown to be equivalent to making the (sometimes suboptimal [7]) choice $\sigma_B = \mathcal{E}_{B|A} \rho_A$ and taking Ω to be the class of operations $\mathbf{LC1}$ (local operations and one-way classical communication from Alice to Bob).

Since the Wang-Renner bound is asymptotically tight for the unassisted capacity (and even for the product state capacity, thus recovering the HSW theorem), the stronger bound obtained using \mathbf{L} also has these properties. Unfortunately, it lacks an SDP formulation and does not possess the concavity property mentioned above. However, the formulation in terms of restricted hypothesis testing makes it clear that by moving to less restrictive conditions on the test, we might obtain weaker, but more tractable bounds. When Ω is $\mathbf{LC1}$, or the larger class \mathbf{PPT} (see next section), the concavity property does hold (see Theorem 22), and we can therefore use the symmetrisation arguments.

For **PPT** the bound has the advantage that it can be formulated as an SDP (see subsection V-A). It seems unlikely that the **PPT** bound is, in general, asymptotically tight, but it might prove useful for certain channels.

IV. PROOF VIA METACONVERSE

Just as in [6], our main results (Theorems 18 and 19) are consequences of a more general ‘meta-converse.’ Following [14], [15] we first express the general idea of the meta-converse using ‘generalised divergences.’ The hypothesis testing based bounds can be obtained by using the hypothesis-testing relative entropies as the divergences.

Let $(1 - \epsilon, \epsilon)_C$ denote a state diagonal in the classical basis with eigenvalues $(1 - \epsilon, \epsilon)$ on a two-dimensional system C (it represents a binary probability distribution).

Suppose that, given a code \mathcal{Z} , we can find a measurement operation $\mathcal{T}_{C|AB}$ with a binary outcome such that, for all channel operations $\mathcal{E}_{B|A}$ from A to B , we have

$$\mathcal{T}_{C|AB} \mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}} = (1 - \epsilon, \epsilon)_C, \quad (47)$$

where ϵ is the error probability of the code \mathcal{Z} for \mathcal{E} , and $\rho_{\tilde{A}\tilde{A}}$ is a canonical purification of the average input ρ_A made to the channel by the encoder (see Definition 14).

Taking a reference channel operation $\mathcal{F}_{B|A}$ for which the error probability f obtained by the code is known (or bounded) and a measure of state distinguishability (a ‘generalised divergence’) \mathbf{d} which is non-increasing under the operation \mathcal{T} we have

$$\mathbf{d}((1 - \epsilon, \epsilon)_C, (1 - f, f)_C) \quad (48)$$

$$= \mathbf{d}(\mathcal{T}_{C|AB} \mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}}, \mathcal{T}_{C|AB} \mathcal{F}_{B|A} \rho_{\tilde{A}\tilde{A}}) \quad (49)$$

$$\leq \mathbf{d}(\mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}}, \mathcal{F}_{B|A} \rho_{\tilde{A}\tilde{A}}), \quad (50)$$

bounding the difference between ϵ and the known quantity f .

In the classical meta-converse of Polyanskiy et al. [6] $\rho_A = \sum_x p(x) |x\rangle\langle x|_A$ is the classical distribution on the input alphabet induced by the encoding of equiprobable messages, and $\mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}}$ is the joint probability distribution over the input and output alphabets induced by the use of the channel. In Wang–Renner and in Hayashi [16], $\mathcal{E}_{B|A}$ is a classical-quantum (c-q) channel which takes input symbol x to some output state $\tau(x)_B$. In these bounds ρ_A is again a probability distribution over the input symbols, while $\mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}}$ is now the quantum state

$$\mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}} = \sum_x p(x) |x\rangle\langle x|_{\tilde{A}} \otimes \tau(x)_B \quad (51)$$

where $\tau(x)_B$ is the output of the classical-quantum channel on the input symbol x . If a code of size M is used with a channel operation whose output is a fixed state σ , independent of its input, its error probability is $1/M$. Hayashi and Wang–Renner implicitly use such an operation for $\mathcal{F}_{B|A}$ with $\sigma_B = \mathcal{E}_{B|A} \rho_A$. In Theorem 27 of Polyanskiy et al. [6] the bound is optimised over all such operations. Hayashi uses the quantum relative Rényi entropy for \mathbf{d} while Wang–Renner and Polyanskiy et al. use the (unrestricted) hypothesis-testing relative entropy, which itself depends on ϵ .

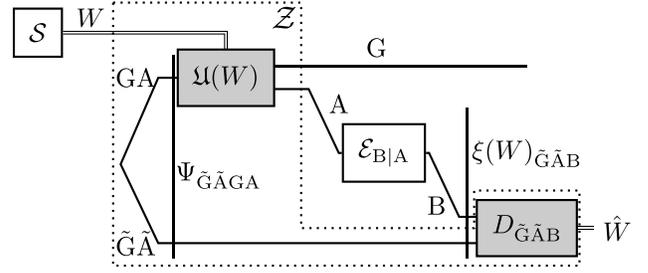


Fig. 3. Reformulation of the protocol of Fig. 1.

We first show how to construct from an entanglement-assisted code a measurement operation satisfying (47).

Proposition 20: From any entanglement-assisted code \mathcal{Z} and source S , such that the average input state is ρ_A and the error probability is ϵ for the channel \mathcal{E} , one can construct a test $T_{\tilde{A}\tilde{B}} \in \mathbf{T}(\tilde{A} : \tilde{B})$ such that, for all $\mathcal{E}_{B|A} \in \mathbf{ops}(A \rightarrow B)$,

$$1 - \epsilon = \Pr(\hat{W} = W | \mathcal{E}, \mathcal{Z}, S) = \text{Tr}_{\tilde{A}\tilde{B}} T_{\tilde{A}\tilde{B}} \mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}}. \quad (52)$$

Furthermore, if \mathcal{Z} is unassisted, then $T_{\tilde{A}\tilde{B}} \in \mathbf{T}^L(\tilde{A} : \tilde{B})$.

Proof: We consider a general entanglement-assisted code as depicted in Figure 1, and use the notation established in Definition 8. Since it is always possible to augment A_E to find a purification $\Psi_{A_E' A_E B_E}$ of $\Psi_{A_E B_E}$, and take $\mathcal{C}(w)_{A|A_E' A_E} := \mathcal{C}(w)_{A|A_E} \text{Tr}_{A_E'}$, we can assume that $\Psi_{A_E B_E}$ is pure in the first place.

Let the isometry $U(w) \in \mathcal{L}(\mathcal{H}_{A_E}, \mathcal{H}_G \otimes \mathcal{H}_A)$ be the Stinespring representation of the encoding map $\mathcal{C}(w)_{A|A_E}$, where G is the discarded environment system. In fact, we can just take $A_E = GA$ so that $U(w)_{GA}$ is a unitary. So, the encoding map for message w is $\mathcal{C}(w)_{A|GA} : X_{GA} \mapsto \text{Tr}_G U(w)_{GA} X_{GA} U(w)_{GA}^\dagger$. Finally, there is no loss of generality in demanding that $B_E = \tilde{G}\tilde{A} \cong GA$. This reformulation of the protocol of Fig. 1 is illustrated in Fig. 3.

First note that

$$U(w)_{GA} \Psi_{GA \tilde{G}\tilde{A}} U(w)_{GA}^\dagger = M(w)_{GA} \Phi_{GA \tilde{G}\tilde{A}} M(w)_{GA}^\dagger \quad (53)$$

where $M(w)_{GA} := U(w)_{GA} \Psi_{GA}^{\frac{1}{2}}$ and $\Psi_{GA} := \text{Tr}_{\tilde{G}\tilde{A}} \Psi_{GA \tilde{G}\tilde{A}}$. If the message W is w then the state of GA after encoding is

$$\rho(w)_{GA} = M(w)_{GA} M(w)_{GA}^\dagger. \quad (54)$$

Referring to the diagram, we see that

$$\Pr(\hat{W} = \hat{w} | W = w, \mathcal{E}, \mathcal{Z}) = \text{Tr}_{\tilde{G}\tilde{A}\tilde{B}} D(\hat{w})_{\tilde{G}\tilde{A}\tilde{B}} \check{\zeta}(w)_{\tilde{G}\tilde{A}\tilde{B}} \quad (55)$$

where $D(\hat{w})_{\tilde{G}\tilde{A}\tilde{B}}$ is the POVM element corresponding to the decoded message being \hat{w} , and where

$$\check{\zeta}(w)_{\tilde{G}\tilde{A}\tilde{B}} = \text{Tr}_G \mathcal{E}_{B|A} (M(w)_{GA} \Phi_{GA \tilde{G}\tilde{A}} M(w)_{GA}^\dagger) \quad (56)$$

$$= \text{Tr}_G M(w)_{GA}^\dagger \Phi_{\tilde{G}\tilde{A}} (\mathcal{E}_{B|A} \Phi_{\tilde{A}\tilde{A}}) M(w)_{GA}^* \quad (57)$$

$$= M(w)_{GA}^\dagger \mathbb{1}_{\tilde{G}} (\mathcal{E}_{B|A} \Phi_{\tilde{A}\tilde{A}}) M(w)_{GA}^* \quad (58)$$

Here we have used the facts noted in Remark 13 and that $\Phi_{\text{GA}\tilde{\text{G}}\tilde{\text{A}}} = \Phi_{\tilde{\text{A}}\tilde{\text{A}}}\Phi_{\tilde{\text{G}}\tilde{\text{G}}}$. The probability of successful decoding is

$$\Pr(\hat{W} = W|\mathcal{E}, \mathcal{Z}, \mathcal{S}) \quad (59)$$

$$= \sum_{w=1}^M \mathcal{S}(w) \text{Tr}_{\tilde{\text{G}}\tilde{\text{A}}\tilde{\text{B}}} D(w)_{\tilde{\text{G}}\tilde{\text{A}}\tilde{\text{B}}} \zeta(w)_{\tilde{\text{G}}\tilde{\text{A}}\tilde{\text{B}}} \quad (60)$$

$$= \text{Tr}_{\tilde{\text{G}}\tilde{\text{A}}\tilde{\text{B}}} R_{\tilde{\text{G}}\tilde{\text{A}}\tilde{\text{B}}} \mathcal{E}_{\text{B}|\text{A}} \Phi_{\tilde{\text{A}}\tilde{\text{A}}} \quad (61)$$

where

$$R_{\tilde{\text{G}}\tilde{\text{A}}\tilde{\text{B}}} := \sum_{w=1}^M \mathcal{S}(w) M(w)_{\tilde{\text{G}}\tilde{\text{A}}}^* D(w)_{\tilde{\text{G}}\tilde{\text{A}}\tilde{\text{B}}} M(w)_{\tilde{\text{G}}\tilde{\text{A}}}^T. \quad (62)$$

Since $R_{\tilde{\text{G}}\tilde{\text{A}}\tilde{\text{B}}}$ is given by a completely positive map (with Kraus operators $\sqrt{\mathcal{S}(w)}M(w)_{\tilde{\text{G}}\tilde{\text{A}}}^*$) acting on $D_{\tilde{\text{G}}\tilde{\text{A}}\tilde{\text{B}}}$, which satisfies $0 \leq D_{\tilde{\text{G}}\tilde{\text{A}}\tilde{\text{B}}} \leq \mathbb{1}_{\tilde{\text{G}}\tilde{\text{A}}\tilde{\text{B}}}$, we have $0 \leq R_{\tilde{\text{G}}\tilde{\text{A}}\tilde{\text{B}}} \leq \rho_{\tilde{\text{G}}\tilde{\text{A}}} \mathbb{1}_{\tilde{\text{B}}}$, where $\rho_{\tilde{\text{G}}\tilde{\text{A}}} := \sum_{w=1}^M \mathcal{S}(w) \rho(w)_{\tilde{\text{G}}\tilde{\text{A}}}$ is the average state of GA after encoding, $\rho_{\tilde{\text{G}}\tilde{\text{A}}\tilde{\text{G}}\tilde{\text{A}}} := \rho_{\tilde{\text{G}}\tilde{\text{A}}}^{1/2} \Phi_{\text{GA}\tilde{\text{G}}\tilde{\text{A}}} \rho_{\tilde{\text{G}}\tilde{\text{A}}}^{1/2}$ is its canonical purification, and $\rho_{\tilde{\text{G}}\tilde{\text{A}}} := \text{Tr}_{\text{GA}} \rho_{\tilde{\text{G}}\tilde{\text{A}}\tilde{\text{G}}\tilde{\text{A}}}$. Here we have used (54) and the ‘transpose trick’ again. Therefore,

$$T_{\tilde{\text{A}}\tilde{\text{B}}} := \rho_{\tilde{\text{A}}}^{-\frac{1}{2}} R_{\tilde{\text{A}}\tilde{\text{B}}} \rho_{\tilde{\text{A}}}^{-\frac{1}{2}} \quad (63)$$

satisfies $0 \leq T_{\tilde{\text{A}}\tilde{\text{B}}} \leq \mathbb{1}_{\tilde{\text{A}}\tilde{\text{B}}}$, and

$$\Pr(\hat{W} = W|\mathcal{E}, \mathcal{Z}, \mathcal{S}) = \text{Tr}_{\tilde{\text{A}}\tilde{\text{B}}} \rho_{\tilde{\text{A}}}^{\frac{1}{2}} (\mathcal{E}_{\text{B}|\text{A}} \Phi_{\tilde{\text{A}}\tilde{\text{A}}}) \rho_{\tilde{\text{A}}}^{\frac{1}{2}} T_{\tilde{\text{A}}\tilde{\text{B}}} \quad (64)$$

$$= \text{Tr}_{\tilde{\text{A}}\tilde{\text{B}}} T_{\tilde{\text{A}}\tilde{\text{B}}} \mathcal{E}_{\text{B}|\text{A}} \rho_{\tilde{\text{A}}\tilde{\text{A}}} \quad (65)$$

as promised.

As noted in the caption for Figure 1, any *unassisted* quantum code corresponds to restricting Bob’s decoding measurement to the output system of the channel, so that

$$\forall w \in \{1, \dots, M\}: D(w)_{\tilde{\text{G}}\tilde{\text{A}}\tilde{\text{B}}} = \mathbb{1}_{\tilde{\text{G}}\tilde{\text{A}}} D(w)_{\tilde{\text{B}}}. \quad (66)$$

Substituting this into (62), we see that

$$T_{\tilde{\text{A}}\tilde{\text{B}}} = \sum_{w=1}^M E(w)_{\tilde{\text{A}}} D(w)_{\tilde{\text{B}}} \quad (67)$$

where the positive operators

$$E(w)_{\tilde{\text{A}}} := S(w) (\rho_{\tilde{\text{A}}})^{-\frac{1}{2}} (\text{Tr}_{\tilde{\text{G}}\tilde{\text{A}}} \rho(w)_{\tilde{\text{G}}\tilde{\text{A}}}) (\rho_{\tilde{\text{A}}})^{-\frac{1}{2}} \quad (68)$$

satisfy

$$\sum_{w=1}^M E(w)_{\tilde{\text{A}}} = \rho_{\tilde{\text{A}}}^{-\frac{1}{2}} \rho_{\tilde{\text{A}}} \rho_{\tilde{\text{A}}}^{-\frac{1}{2}} \leq \mathbb{1}_{\tilde{\text{A}}}. \quad (69)$$

Letting $E(0)_{\tilde{\text{A}}} := \mathbb{1}_{\tilde{\text{A}}} - \rho_{\tilde{\text{A}}}^{-\frac{1}{2}} \rho_{\tilde{\text{A}}} \rho_{\tilde{\text{A}}}^{-\frac{1}{2}}$, the operators $E(0)_{\tilde{\text{A}}}, \dots, E(M)_{\tilde{\text{A}}}$ constitute a POVM, and so, for an unassisted code, $T_{\tilde{\text{A}}\tilde{\text{B}}}$ is a *local* test i.e. $T_{\tilde{\text{A}}\tilde{\text{B}}} \in \mathbf{T}^{\text{L}}(\tilde{\text{A}} : \tilde{\text{B}})$. The local implementation is simply this: Alice performs a measurement with POVM elements $E(w)_{\tilde{\text{A}}}$ and Bob performs the decoding measurement (with POVM elements $D(w)_{\tilde{\text{B}}}$). If their outcomes are equal then the test accepts, the ‘accepting’ POVM element being given in equation (67). ■

A quantum generalisation of the ‘meta-converse’ [6, Th. 26], is now straightforward:

Proposition 21 (Meta-Converse): Let \mathcal{Z} be an entanglement-assisted code which, when used with \mathcal{S} ,

induces the average input state ρ_{A} , and which has success probability

$$\Pr(\hat{W} = W|\mathcal{E}^{(i)}, \mathcal{Z}, \mathcal{S}) = 1 - \epsilon_i, \quad (70)$$

when used with channel operation $\mathcal{E}_{\text{B}|\text{A}}^{(i)}$, for $i \in \{0, 1\}$. Consider the hypothesis testing problem where H_i asserts that the state of $\tilde{\text{A}}\tilde{\text{B}}$ is $\mathcal{E}_{\text{B}|\text{A}}^{(i)} \rho_{\tilde{\text{A}}\tilde{\text{A}}}$. If we accept H_0 when the test constructed from $(\mathcal{Z}, \mathcal{S})$ as in Proposition 20 accepts, then

$$\beta = \text{Tr}_{\tilde{\text{A}}\tilde{\text{B}}} T_{\tilde{\text{A}}\tilde{\text{B}}} \mathcal{E}_{\text{B}|\text{A}}^{(1)} \rho_{\tilde{\text{A}}\tilde{\text{A}}} = 1 - \epsilon_1 \quad (71)$$

and

$$1 - \alpha = \text{Tr}_{\tilde{\text{A}}\tilde{\text{B}}} T_{\tilde{\text{A}}\tilde{\text{B}}} \mathcal{E}_{\text{B}|\text{A}}^{(0)} \rho_{\tilde{\text{A}}\tilde{\text{A}}} = 1 - \epsilon_0. \quad (72)$$

Therefore, by definition,

$$\beta_{\epsilon_0}^{\text{ALL}}(\mathcal{E}_{\text{B}|\text{A}}^{(0)} \rho_{\tilde{\text{A}}\tilde{\text{A}}} \| \mathcal{E}_{\text{B}|\text{A}}^{(1)} \rho_{\tilde{\text{A}}\tilde{\text{A}}}) \leq 1 - \epsilon_1. \quad (73)$$

Furthermore, if constraints on the code \mathcal{Z} mean that $T_{\tilde{\text{A}}\tilde{\text{B}}}$ is guaranteed to belong to some class of tests $\mathbf{T}^{\Omega}(\tilde{\text{A}} : \tilde{\text{B}})$, then the (potentially more stringent) bound

$$\beta_{\epsilon_0}^{\Omega}(\mathcal{E}_{\text{B}|\text{A}}^{(0)} \rho_{\tilde{\text{A}}\tilde{\text{A}}} \| \mathcal{E}_{\text{B}|\text{A}}^{(1)} \rho_{\tilde{\text{A}}\tilde{\text{A}}}) \leq 1 - \epsilon_1. \quad (74)$$

also applies.

We now apply the meta-converse to obtain our main results. Suppose that $\mathcal{E}_{\text{B}|\text{A}}^{(1)}$ is a completely useless channel operation which has $\mathcal{E}_{\text{B}|\text{A}}^{(1)} \rho_{\text{A}} = \sigma_{\text{B}}$ for all ρ_{A} , and assume that the M messages are equiprobable i.e. $\mathcal{S} = S_M$. Then, it is easily verified that $1 - \epsilon_1 = 1/M$ (in fact any other value would imply that communication is possible in the absence of a channel.) Setting $\mathcal{E}_{\text{B}|\text{A}}^{(0)} = \mathcal{E}_{\text{B}|\text{A}}$, the hypothesis testing problem described in the Proposition above is now exactly the one shown in Figure 2. With these choices, equation (74) tells us that any $(M, \epsilon, \rho_{\text{A}})$ code whose corresponding test belongs to $\mathbf{T}^{\Omega}(\tilde{\text{A}} : \tilde{\text{B}})$ must satisfy

$$\max_{\sigma_{\text{B}}} \beta_{\epsilon}^{\Omega}(\mathcal{E}_{\text{B}|\text{A}} \rho_{\tilde{\text{A}}\tilde{\text{A}}} \| \rho_{\tilde{\text{A}}} \sigma_{\text{B}}) \leq 1/M. \quad (75)$$

Here, we maximise over σ_{B} to obtain the best possible bound. For entanglement-assisted codes, rearranging this and using Definition 3 gives us

$$\log M_{\epsilon}^{\text{E}}(\mathcal{E}_{\text{B}|\text{A}}, \rho_{\text{A}}) \leq \min_{\sigma_{\text{B}}} D_{\epsilon}^{\text{ALL}}(\mathcal{E}_{\text{B}|\text{A}} \rho_{\tilde{\text{A}}\tilde{\text{A}}} \| \rho_{\tilde{\text{A}}} \sigma_{\text{B}}) \quad (76)$$

(Theorem 18) and for unassisted codes, we can write the stronger bound

$$\log M_{\epsilon}(\mathcal{E}_{\text{B}|\text{A}}, \rho_{\text{A}}) \leq \min_{\sigma_{\text{B}}} D_{\epsilon}^{\text{L}}(\mathcal{E}_{\text{B}|\text{A}} \rho_{\tilde{\text{A}}\tilde{\text{A}}} \| \rho_{\tilde{\text{A}}} \sigma_{\text{B}}) \quad (77)$$

(which is Theorem 19).

V. PROPERTIES OF THE BOUNDS

In this section, we generalise results of Polyanskiy [7], showing that $\beta_{\epsilon}^{\Omega}(\mathcal{E}_{\text{B}|\text{A}} \rho_{\tilde{\text{A}}\tilde{\text{A}}} \| \rho_{\tilde{\text{A}}} \sigma_{\text{B}})$ is convex in ρ_{A} and concave in σ_{B} , provided Ω contains **LC1**. This enables one to use symmetries of the channel to restrict the optimisation over ρ_{A} and σ_{B} to states with corresponding symmetries, as we show in subsection V-E.

Let $\beta(T, \rho_{\tilde{\text{A}}}, \sigma_{\text{B}}) := \text{Tr}_{\tilde{\text{A}}\tilde{\text{B}}} T_{\tilde{\text{A}}\tilde{\text{B}}} \rho_{\tilde{\text{A}}} \sigma_{\text{B}}$. This is a bilinear function of T and σ_{B} . Therefore, the minimum of $\beta(T, \rho_{\tilde{\text{A}}}, \sigma_{\text{B}})$

over T is concave in σ_B , and since the set **states**(B) and the set of tests satisfying $\alpha(T, \rho_{\tilde{A}}) := 1 - \text{Tr}_{\tilde{A}B} T_{\tilde{A}B} \mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}} \leq \epsilon$ are both convex, von Neumann's minimax theorem tells us that $\max_{\sigma_B} \beta_{\epsilon}^{\Omega}(\mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}}, \rho_{\tilde{A}} \sigma_B)$ is equal to

$$\min_{T_{\tilde{A}B} \in \Omega} \beta^*(T) \quad (78)$$

$$\begin{aligned} &\text{subject to} \\ &\text{Tr}_{\tilde{A}B} T_{\tilde{A}B} \mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}} \geq 1 - \epsilon, \end{aligned} \quad (79)$$

where

$$\beta^*(T, \rho_{\tilde{A}}) := \max_{\sigma_B} \text{Tr}_{\tilde{A}B} T_{\tilde{A}B} \rho_{\tilde{A}} \sigma_B \quad (80)$$

$$= \|\text{Tr}_A T_{\tilde{A}B} \rho_{\tilde{A}}\|_{\infty} \quad (81)$$

As noted in [7] for the classical case, $\beta^*(T)$ is the (worst case) probability of type II error for $T_{\tilde{A}B}$ in the *compound* hypothesis testing problem where H_0 is still that the state is $\mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}}$, but now H_1 is the compound hypothesis that the state belongs to the set $\{\rho_{\tilde{A}} \sigma_B : \sigma_B \in \mathbf{states}(B)\}$.

Theorem 22: For any operations $\mathcal{E}_{B|A}^{(0)}$ and $\mathcal{E}_{B|A}^{(1)}$, and class of bipartite operations Ω which contains **LCl**, the function

$$(\epsilon, \rho_A) \mapsto \beta_{\epsilon}^{\Omega}(\mathcal{E}_{B|A}^{(0)} \rho_{\tilde{A}\tilde{A}} \| \mathcal{E}_{B|A}^{(1)} \rho_{\tilde{A}\tilde{A}}) \quad (82)$$

is jointly convex in ϵ and ρ_A .

Proof: Suppose that we have, for all $j \in \{1, \dots, m\}$, $\epsilon_j \in [0, 1]$, $\rho_A^{(j)} \in \mathbf{states}(A)$, and $\lambda_j \geq 0$ such that $\sum_{j=1}^m \lambda_j = 1$, $\sum_{j=1}^m \lambda_j \rho_A^{(j)} = \rho_A$, and $\sum_{j=1}^m \lambda_j \epsilon_j = \epsilon$.

Let $T(j)_{\tilde{A}B}$ be a test in $\mathbf{T}^{\Omega}(\tilde{A} : B)$ that achieves $\beta_j := \beta_{\epsilon_j}^{\Omega}(\mathcal{E}_{B|A}^{(0)} \rho_{\tilde{A}\tilde{A}}^{(j)} \| \mathcal{E}_{B|A}^{(1)} \rho_{\tilde{A}\tilde{A}}^{(j)})$. That is to say,

$$\text{Tr}_{\tilde{A}B} (\mathbb{1} - T(j))_{\tilde{A}B} \mathcal{E}_{B|A}^{(0)} \rho_{\tilde{A}\tilde{A}}^{(j)} \leq \epsilon_j, \quad (83)$$

$$\text{and } \text{Tr}_{\tilde{A}B} T(j)_{\tilde{A}B} \mathcal{E}_{B|A}^{(1)} \rho_{\tilde{A}\tilde{A}}^{(j)} = \beta_j. \quad (84)$$

The claim is that

$$\beta_{\epsilon}^{\Omega}(\mathcal{E}_{B|A}^{(0)} \rho_{\tilde{A}\tilde{A}} \| \mathcal{E}_{B|A}^{(1)} \rho_{\tilde{A}\tilde{A}}) \leq \sum_{j=1}^m \lambda_j \beta_j. \quad (85)$$

We shall explicitly construct a test in Ω which demonstrates (85). Consider the suboperations

$$\mathcal{M}_{\tilde{A}|\tilde{A}}^{(j)} : X_{\tilde{A}} \mapsto \lambda_j \rho_{\tilde{A}}^{(j)1/2} \rho_{\tilde{A}}^{-1/2} X_{\tilde{A}} \rho_{\tilde{A}}^{-1/2} \rho_{\tilde{A}}^{(j)1/2} \quad (86)$$

for $j \in \{1, \dots, m\}$ and

$$\mathcal{M}_{\tilde{A}|\tilde{A}}^{(0)} : X_{\tilde{A}} \mapsto \Pi_{\tilde{A}} X_{\tilde{A}} \Pi_{\tilde{A}} \quad (87)$$

where $\Pi_{\tilde{A}}$ is the orthogonal projector onto the kernel of $\rho_{\tilde{A}}$. The key property of these suboperations is that

$$\mathcal{M}_{\tilde{A}|\tilde{A}}^{(j)} \rho_{\tilde{A}\tilde{A}} = \begin{cases} \lambda_j \rho_{\tilde{A}\tilde{A}}^{(j)} & \text{for } j \in \{1, \dots, m\}, \\ 0 & \text{for } j = 0. \end{cases} \quad (88)$$

Since

$$\Pi_{\tilde{A}} + \sum_{j=1}^m \lambda_j \rho_{\tilde{A}}^{-1/2} \rho_{\tilde{A}}^{(j)} \rho_{\tilde{A}}^{-1/2} = \mathbb{1}_{\tilde{A}}, \quad (89)$$

the suboperations $\{\mathcal{M}_{\tilde{A}|\tilde{A}}^{(j)}\}_{j=0}^m$ constitute an instrument.

Suppose that Alice performs this instrument, and classically communicates the outcome to Bob. If the outcome is j then they perform the test with POVM element $T(j)_{\tilde{A}B}$ corresponding to deciding on hypothesis 0. If their decision is stored in a register C, then the overall measurement operation is

$$\mathcal{M}_{C|\tilde{A}B} = \sum_{j=0}^m \{|0\rangle\langle 0|_C \text{Tr}_{\tilde{A}B} T_{\tilde{A}B}^{(j)} + |1\rangle\langle 1|_C \text{Tr}_{\tilde{A}B} (\mathbb{1} - T^{(j)})_{\tilde{A}B}\} \mathcal{M}_{\tilde{A}|\tilde{A}}^{(j)}. \quad (90)$$

$$\quad (91)$$

Since this implementation uses only one-way classical communication followed by operations in class Ω , which, by hypothesis, contains **LCl**, this test is also in $\mathbf{T}^{\Omega}(\tilde{A} : B)$. Using (88) and (83), we find that the type I error of this test is

$$\text{Tr}_C |1\rangle\langle 1|_C \mathcal{M}_{C|\tilde{A}B} \mathcal{E}_{B|A}^{(0)} \rho_{\tilde{A}\tilde{A}} = \sum_{j=1}^m \lambda_j \epsilon_j \leq \epsilon \quad (92)$$

while its type II error is

$$\text{Tr}_C |0\rangle\langle 0|_C \mathcal{M}_{C|\tilde{A}B} \mathcal{E}_{B|A}^{(1)} \rho_{\tilde{A}\tilde{A}} = \sum_{j=1}^m \lambda_j \beta_j, \quad (93)$$

and so the claimed convexity (85) does hold. \blacksquare

Corollary 23: $\beta_{\epsilon}^{\Omega}(\mathcal{E}_{B|A}, \rho_A)$ is also jointly convex in ϵ and ρ_A , as it is given by maximising over functions with this property.

A. Semidefinite Programs

As mentioned in the introduction, the converse for entanglement-assisted codes in terms of unrestricted hypothesis testing, and the converse for unassisted codes in terms of PPT hypothesis testing, can be formulated as semidefinite programs. Here we give the first of these in detail, and describe how to add the PPT constraint.

Letting $R_{\tilde{A}B} := \rho_{\tilde{A}}^{1/2} T_{\tilde{A}B} \rho_{\tilde{A}}^{1/2}$, we have

$$\beta^*(T, \rho_{\tilde{A}}) = \|\text{Tr}_A R_{\tilde{A}B}\|_{\infty} \quad (94)$$

$$= \min\{\lambda : \lambda \mathbb{1}_B \geq \text{Tr}_A R_{\tilde{A}B}\}, \quad (95)$$

and $\alpha(T, \rho_{\tilde{A}}) = \text{Tr}_{\tilde{A}B} R_{\tilde{A}B} \mathcal{E}_{B|A} \Phi_{\tilde{A}\tilde{A}}$.

Proposition 24 (Primal SDP):

$$I_{\epsilon}^{\text{ALL}}(\mathcal{E}_{B|A}, \rho_A) = -\log \min \lambda \quad (96)$$

subject to

$$\text{Tr}_A R_{\tilde{A}B} \leq \lambda \mathbb{1}_B, \quad (97)$$

$$\text{Tr}_{\tilde{A}B} R_{\tilde{A}B} \mathcal{E}_{B|A} \Phi_{\tilde{A}\tilde{A}} \geq 1 - \epsilon, \quad (98)$$

$$R_{\tilde{A}B} \leq \rho_{\tilde{A}} \mathbb{1}_B, \quad (99)$$

$$R_{\tilde{A}B} \geq 0. \quad (100)$$

Since the constraints on ρ_A are semidefinite, the bound

$$\max_{\rho_A \in \mathcal{D}(\mathcal{H}_A)} I_{\epsilon}^{\text{ALL}}(\mathcal{E}_{B|A}, \rho_A) \quad (101)$$

from Theorem 18 is also a semidefinite program.

Remark 25: The constraint $0 \leq \text{t}_{B|B} T_{\tilde{A}B} \leq \mathbb{1}$ (where t_B is the transpose map on system B) is equivalent to

$$0 \leq \text{t}_{B|B} R_{\tilde{A}B} \leq \rho_{\tilde{A}} \mathbb{1}_B. \quad (102)$$

Because the transpose map is linear, adding these constraints on $R_{\tilde{A}B}$ to the primal SDP above yields an SDP for $I_\epsilon^{\text{PPT}}(\mathcal{E}_{B|A}, \rho_A)$.

Associating operators $F_{\tilde{A}B}$ and G_B with constraints (99) and (97), and a real multiplier μ with (98) yields the Lagrangian

$$\lambda + \text{Tr}_B G_B (\text{Tr}_{\tilde{A}} R_{\tilde{A}B} - \lambda \mathbb{1}_B) \quad (103)$$

$$+ \text{Tr}_{\tilde{A}B} F_{\tilde{A}B} (R_{\tilde{A}B} - \rho_{\tilde{A}} \mathbb{1}_B) \quad (104)$$

$$+ \mu(1 - \epsilon - \text{Tr} R_{\tilde{A}B} \mathcal{E}_{B|A} \Phi_{\tilde{A}A}) \quad (105)$$

$$= \text{Tr}_{\tilde{A}B} R_{\tilde{A}B} (\mathbb{1}_{\tilde{A}} G_B + F_{\tilde{A}B} - \mathcal{E}_{B|A} \Phi_{\tilde{A}A}) \quad (106)$$

$$+ \lambda(1 - \text{Tr}_B G_B) + (1 - \epsilon)\mu \quad (107)$$

$$- \text{Tr}_{\tilde{A}B} F_{\tilde{A}B} \rho_{\tilde{A}} \mathbb{1}_B \quad (108)$$

from which one can derive the dual SDP. Below, we show that the optimal value of this dual SDP is equal to the optimal value of the primal.

Proposition 26 (Dual SDP):

$$I_\epsilon^{\text{ALL}}(\mathcal{E}_{B|A}, \rho_A) = -\log\{\max(1 - \epsilon)\mu - \text{Tr} F_{\tilde{A}} \rho_{\tilde{A}}\} \quad (109)$$

subject to

$$\mathbb{1}_{\tilde{A}} G_B + F_{\tilde{A}B} \geq \mu \mathcal{E}_{B|A} \Phi_{\tilde{A}A}, \quad (110)$$

$$\text{Tr}_B G_B \leq 1, \quad (111)$$

$$G_B, F_{\tilde{A}B}, \mu \geq 0, \quad (112)$$

where $F_{\tilde{A}} := \text{Tr}_B F_{\tilde{A}B}$.

Proof: For sufficiently large a , the point given by $G_B = \mathbb{1}_B / (2 \dim(B))$, $F_{\tilde{A}B} = a \mathbb{1}_{\tilde{A}B}$, and any $\mu > 0$ strictly satisfies the dual constraints (110-112), so the dual SDP is strictly feasible, and therefore its solution is equal to the primal solution (see [17, Th. 3.1]). ■

The maximisation of (109) over states ρ_A of \tilde{A} can also be formulated as an SDP, in a similar way to the primal.

B. Classical Channels

Let $\mathcal{C}_{A|A}$ and $\mathcal{C}_{B|B}$ denote the completely dephasing operations in the classical bases for A and B, respectively. A channel operation $\mathcal{E}_{B|A}$ is classical if $\mathcal{E}_{B|A} \mathcal{C}_{A|A} = \mathcal{E}_{B|A}$, and $\mathcal{C}_{B|B} \mathcal{E}_{B|A} = \mathcal{E}_{B|A}$, and we can therefore restrict the minimization over average input states in our bounds to states ρ_A which are diagonal in the classical basis, thus

$$\rho_A = \sum_x p(x) |x\rangle\langle x|_A. \quad (113)$$

Furthermore, since the state $\mathcal{E}_{B|A} \rho_{\tilde{A}A}$ is invariant under the operation $\mathcal{C}_{\tilde{A}|\tilde{A}} \mathcal{C}_{B|B} \in \text{ops}^L(\tilde{A} : B \rightarrow \tilde{A} : B)$, for any class Ω containing L we have (by Proposition 4)

$$D_\epsilon^\Omega(\mathcal{E}_{B|A} \rho_{\tilde{A}A} \| \rho_{\tilde{A}} \sigma_B) \quad (114)$$

$$\geq D_\epsilon^\Omega(\mathcal{C}_{B|B} \mathcal{E}_{B|A} \rho_{\tilde{A}A} \| \mathcal{C}_{B|B} \rho_{\tilde{A}} \sigma_B) \quad (115)$$

$$= D_\epsilon^\Omega(\mathcal{E}_{B|A} \rho_{\tilde{A}A} \| \rho_{\tilde{A}} \sigma'_B) \quad (116)$$

where σ'_B is diagonal in the classical basis of B. Therefore, we can also restrict the optimisation over σ_B to classical states and, by the discussion in subsection II-B, restrict the test

POVM element to be diagonal in the classical basis. Therefore, Theorems 18 and 19 both reduce to the PPV converse (for finite alphabets) when the channel is classical.

C. Comparison With Wang–Renner

In our notation, the Wang–Renner converse states that for c-q channels with finite input alphabet A and output states $\tau(x)_B$ for $x \in A$

$$\log M_\epsilon \leq \sup_p D_\epsilon^{\text{ALL}}(\tau_{CB} \| \tau_C \tau_B) \quad (117)$$

where C is a system of dimension $|A|$ and $\tau_{CB} := \sum_{x \in A} p(x) |x\rangle\langle x|_C \otimes \tau(x)_B$. To apply their converse to general channels, one notes that any (unassisted) code for a general quantum channel, induces a c-q channel by its specification of the input states used in the code. Together with the choice of p this yields a c-q state

$$\tau_{CB} = \sum_x p(x) |x\rangle\langle x|_C \otimes \mathcal{E}_{B|A} \rho(x)_A. \quad (118)$$

Optimising the Wang–Renner bound over all choices of input states and distributions p such that the average channel input to the quantum channel is ρ_A , one obtains

$$M_\epsilon(\mathcal{E}_{B|A}) \leq \chi_\epsilon(\mathcal{E}_{B|A}, \rho_A) \quad (119)$$

where

Definition 27:

$$\chi_\epsilon(\mathcal{E}_{B|A}, \rho_A) := \max_{\eta_{CA} \in \text{Ens}(\rho_A)} D_\epsilon^{\text{ALL}}(\mathcal{E}_{B|A} \eta_{CA} \| \eta_C \mathcal{E}_{B|A} \rho_A), \quad (120)$$

where $\text{Ens}(\rho_A)$ is the set of all states of the form

$$\sum_{k=1}^{\dim(C)} p_k |k\rangle\langle k|_C \otimes \rho(k)_A \quad (121)$$

where C is some finite dimensional system (acting as a classical register) and where $p_k \geq 0$, $\sum_k p_k = 1$, $\rho(k)_A \in \text{states}(A)$, and $\sum_k p_k \rho(k)_A = \rho_A$.

This notation is motivated by the fact that the Holevo bound [18] is given by $\chi(\mathcal{E}_{B|A}, \rho_A) = \max_{\eta_{CA} \in \text{Ens}(\rho_A)} D(\mathcal{E}_{B|A} \eta_{CA} \| \eta_C \mathcal{E}_{B|A} \rho_A)$, where D is the usual quantum relative entropy [19].

Proposition 28:

$$\chi_\epsilon(\mathcal{E}_{B|A}, \rho_A) = D_\epsilon^{\text{LC1}}(\mathcal{E}_{B|A} \rho_{\tilde{A}A} \| \rho_{\tilde{A}} \mathcal{E}_{B|A} \rho_A) \quad (122)$$

Proof: Let Alice's measurement have the POVM elements $F(k)_{\tilde{A}}$ where k labels the outcome which she sends to Bob. There is no loss of generality in having Alice perform her measurement before Bob does anything, and storing the outcome in a classical register C to which Bob has access.

Let $p(k) := \text{Tr}_{\tilde{A}} \rho_{\tilde{A}} F(k)_{\tilde{A}}$ be the probability of outcome k and $\rho(k)_A := \text{Tr}_{\tilde{A}} \rho_{\tilde{A}A} F_{\tilde{A}}(k) / p(k)$. Under hypothesis 0 the state of CB is $\mathcal{E}_{B|A} \eta_{CA}$, where

$$\eta_{CA} = \sum_k p(k) |k\rangle\langle k|_C \otimes \rho(k)_A, \quad (123)$$

while under hypothesis 1 the state is $\eta_C \mathcal{E}_{B|A} \eta_A$. Clearly $\eta_A = \sum_k p(k) \rho(k)_A = \rho_A$. Therefore,

$$D_\epsilon^{\text{LCI}}(\mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}} \| \rho_{\tilde{A}} \mathcal{E}_{B|A} \rho_A) \quad (124)$$

$$= \max_{\eta_{CA} \in \text{Ens}(\rho_A)} D_\epsilon^{\text{ALL}}(\mathcal{E}_{B|A} \eta_{CA} \| \eta_C \mathcal{E}_{B|A} \rho_A) \quad (125)$$

$$= \chi_\epsilon(\mathcal{E}_{B|A}, \rho_A). \quad (126)$$

■

Corollary 29: From the above proposition and the definitions of the quantities involved, the inequalities

$$I_\epsilon^{\text{L}}(\mathcal{E}_{B|A}, \rho_A) \leq I_\epsilon^{\text{LCI}}(\mathcal{E}_{B|A}, \rho_A) \leq \chi_\epsilon(\mathcal{E}_{B|A}, \rho_A) \quad (127)$$

follow immediately.

D. Asymptotics

It was already noted in [2] that the (asymptotically tight) Holevo bound on unassisted codes can be recovered from an asymptotic analysis of the Wang-Renner bound, which our bounds on unassisted codes subsume (in fact, an argument of [5] can be used to show that the converse part of the HSW theorem [20], [21] can also be derived).

For entanglement-assisted coding over memoryless quantum channels, Shannon's noisy channel coding theorem has a beautiful generalisation due to Bennett, Shor, Smolin and Thapliyal:

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log M_\epsilon^{\text{E}}((\mathcal{E}^{\otimes n})_{B^n|A^n}) = \max_{\rho_A} I(\mathcal{E}_{B|A}, \rho_A) \quad (128)$$

where

$$I(\mathcal{E}_{B|A}, \rho_A) := S(\rho_A) + S(\mathcal{E}_{B|A}(\rho_A)) - S(\mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}}) \quad (129)$$

is the *quantum mutual information* between systems \tilde{A} and B when the state of $\tilde{A}B$ is $\mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}}$. As noted, for classical channels Theorem 18 reduces to [6, Th. 27]. In [6, Sec. III.G] it is shown how to derive a Fano-type converse from their Theorem 27. The derivation and result generalise perfectly to the entanglement-assisted codes for quantum channels: As usual, the binary entropy is $h(p) := -(1-p) \log(1-p) - p \log p$, and the binary relative entropy is

$$d(p\|q) := D((p, 1-p) \| (q, 1-q)) \quad (130)$$

$$= p \log \frac{p}{q} + (1-p) \log \frac{1-p}{1-q} \quad (131)$$

$$\geq p \log \frac{1}{q} - h(p). \quad (132)$$

By the data processing inequality for quantum relative entropy under CPTP maps, and (132)

$$D(\rho_0 \| \rho_1) \geq d(1 - \epsilon \| \beta_\epsilon(\rho_0, \rho_1)) \quad (133)$$

$$\geq (1 - \epsilon) \log \frac{1}{\beta_\epsilon(\rho_0, \rho_1)} - h(\epsilon). \quad (134)$$

Therefore,

$$D_\epsilon^{\text{ALL}}(\rho_0 \| \rho_1) \leq (D(\rho_0 \| \rho_1) + h(\epsilon)) / (1 - \epsilon). \quad (135)$$

Setting $\rho_0 = \mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}}$ and $\rho_1 = \rho_{\tilde{A}} \sigma_B$, and minimizing over σ_B yields.

Lemma 30:

$$I_\epsilon^{\text{ALL}}(\mathcal{E}_{B|A}, \rho_A) \leq (I(\mathcal{E}_{B|A}, \rho_A) + h(\epsilon)) / (1 - \epsilon). \quad (136)$$

The converse part of (128) is now easily derived: From Theorem 18 and the preceding lemma we have

$$\begin{aligned} \log M_\epsilon^{\text{E}}((\mathcal{E}^{\otimes n})_{B^n|A^n}, \rho_{A^n}) \\ \leq \max_{\rho_{A^n}} \frac{I((\mathcal{E}^{\otimes n})_{B^n|A^n}, \rho_{A^n}) + h(\epsilon)}{1 - \epsilon}. \end{aligned} \quad (137)$$

In [22] Adami and Cerf show that

$$I(\mathcal{E}_{B_1|A_1}^{(1)} \mathcal{E}_{B_2|A_2}^{(2)}, \rho_{A_1 A_2}) = I(\mathcal{E}_{B_1|A_1}^{(1)}, \rho_{A_1}) + I(\mathcal{E}_{B_2|A_2}^{(2)}, \rho_{A_2}), \quad (138)$$

so we have

$$\max_{\rho_{A^n}} I((\mathcal{E}^{\otimes n})_{B^n|A^n}, \rho_{A^n}) \leq n \max_{\rho_A} I(\mathcal{E}_{B|A}, \rho_A), \quad (139)$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log M_\epsilon((\mathcal{E}^{\otimes n})_{B^n|A^n}) \leq \frac{1}{1 - \epsilon} \max_{\rho_A} I(\mathcal{E}_{B|A}, \rho_A). \quad (140)$$

Taking the limit $\epsilon \rightarrow 0$ completes the proof.

E. Using Symmetries

We now show how symmetries of $\mathcal{E}_{B|A}$ can be used to simplify the computation of $I_\epsilon^\Omega(\mathcal{E}_{B|A}, \rho_A)$. For classical channels, analogous results were obtained in [7] (but note that [7] also deals with infinite input/output alphabets and similar ideas were discussed in [13]).

Suppose that there is a group G with a representation $g \mapsto g_{A|A} \in \mathbf{ops}(A \rightarrow A)$ for all $g \in G$, given by

$$g_{A|A} \tau_A = U(g)_A \tau_A U(g)_A^\dagger, \quad (141)$$

and a representation $g \mapsto g_{B|B} \in \mathbf{ops}(B \rightarrow B)$, given by

$$g_{B|B} \tau_B = V(g)_B \tau_B V(g)_B^\dagger, \quad (142)$$

where U and V are unitary representations of G , and let

$$g_{\tilde{A}|\tilde{A}} \tau_{\tilde{A}} := U(g)_A^* \tau_{\tilde{A}} U(g)_A^T. \quad (143)$$

Suppose that $\mathcal{E}_{B|A}$ possesses the G -covariance

$$\forall g \in G : \mathcal{E}_{B|A} g_{A|A} = g_{B|B} \mathcal{E}_{B|A}. \quad (144)$$

Proposition 31: For any $\Omega \supseteq \mathbf{L}$ and for all $g \in G$,

$$\begin{aligned} D_\epsilon^\Omega(\mathcal{E}_{B|A} g_{\tilde{A}|\tilde{A}} g_{A|A} \rho_{\tilde{A}\tilde{A}} \| g_{\tilde{A}|\tilde{A}} g_{B|B} \rho_{\tilde{A}} \sigma_B) \\ = D_\epsilon^\Omega(\mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}} \| \rho_{\tilde{A}} \sigma_B) \end{aligned} \quad (145)$$

and $I_\epsilon^\Omega(\mathcal{E}_{B|A}, g_{A|A} \rho_A) = I_\epsilon^\Omega(\mathcal{E}_{B|A}, \rho_A)$.

Proof: The first claim follows from

$$\mathcal{E}_{B|A} g_{\tilde{A}|\tilde{A}} g_{A|A} \rho_{\tilde{A}\tilde{A}} \quad (146)$$

$$= \mathcal{E}_{B|A} [U(g)_A \rho_{\tilde{A}}^{\frac{1}{2}} U(g)_A^* \Phi_{\tilde{A}\tilde{A}} U(g)_A^T \rho_{\tilde{A}}^{\frac{1}{2}} U(g)_A^\dagger] \quad (147)$$

$$= V(g)_B U(g)_A^* \mathcal{E}_{B|A} [\rho_{\tilde{A}}^{\frac{1}{2}} \Phi_{\tilde{A}\tilde{A}} \rho_{\tilde{A}}^{\frac{1}{2}}] V(g)_B^\dagger U(g)_A^T \quad (148)$$

$$= g_{\tilde{A}|\tilde{A}} g_{B|B} \mathcal{E}_{B|A} \rho_{\tilde{A}\tilde{A}}, \quad (149)$$

the fact that $g_{\tilde{A}|\tilde{A}}g_{B|B}$ and its inverse belong to $L(\tilde{A} : B \rightarrow \tilde{A} : B) \subseteq \Omega$, and Corollary 5. We use this to prove the second claim thus:

$$I_\epsilon^\Omega(\mathcal{E}_{B|A}, \rho_A) \quad (150)$$

$$= \min_{\sigma_B} D_\epsilon^\Omega(\mathcal{E}_{B|A}\rho_{\tilde{A}\tilde{A}} \|\rho_{\tilde{A}}\sigma_B) \quad (151)$$

$$= D_\epsilon^\Omega(\mathcal{E}_{B|A}\rho_{\tilde{A}\tilde{A}} \|\rho_{\tilde{A}}\sigma_B^0) \quad (152)$$

$$= D_\epsilon^\Omega(\mathcal{E}_{B|A}g_{\tilde{A}|\tilde{A}}g_{A|A}\rho_{\tilde{A}\tilde{A}} \|\rho_{\tilde{A}}g_{\tilde{A}|\tilde{A}}g_{B|B}\rho_{\tilde{A}}\sigma_B^0) \quad (153)$$

$$\geq \min_{\sigma_B} D_\epsilon^\Omega(\mathcal{E}_{B|A}g_{\tilde{A}|\tilde{A}}g_{A|A}\rho_{\tilde{A}\tilde{A}} \|\rho_{\tilde{A}}\sigma_B) \quad (154)$$

$$= I_\epsilon^\Omega(\mathcal{E}_{B|A}, g_{A|A}\rho_A). \quad (155)$$

Here we are using the fact that the standard purification of $g_{A|A}\rho_A$ is $g_{\tilde{A}|\tilde{A}}g_{A|A}\rho_{\tilde{A}\tilde{A}}$. Since g has an inverse in G , the reverse inequality holds too. ■

Suppose that there is a Haar (G -invariant) measure μ on G , and let $\bar{\rho}_A := \int_G d\mu(g)g_{A|A}\rho_A$. Then, $\bar{\rho}_A$ is invariant under the action $g_{A|A}$, and by Jensen's inequality, Corollary 23, and Proposition 31,

$$\beta_\epsilon^\Omega(\mathcal{E}_{B|A}, \bar{\rho}_A) \leq \int_G d\mu(g)\beta_\epsilon^\Omega(\mathcal{E}_{B|A}, g_{A|A}\rho_A) \quad (156)$$

$$= \beta_\epsilon^\Omega(\mathcal{E}_{B|A}, \rho_A). \quad (157)$$

Therefore, the optimisation over ρ_A can be restricted to those density operators invariant under the action of $g_{A|A}$.

An important type of symmetry that an operation representing n channel uses may possess is permutation covariance. For example, this applies to n uses of any memoryless channel.

For any element π of the symmetric group S_n , and n -partite system $Q^n := Q_1 Q_2 \dots Q_n$ consisting of n isomorphic systems Q_j , let $\pi_{Q^n|Q^n} \in \mathbf{ops}(Q^n \rightarrow Q^n)$ denote the unitary operation which permutes the n systems.

An operation $\mathcal{E}_{B^n|A^n} \in \mathbf{ops}(A^n \rightarrow B^n)$ is permutation covariant if

$$\mathcal{E}_{B^n|A^n}\pi_{A^n|A^n} = \pi_{B^n|B^n}\mathcal{E}_{B^n|A^n}. \quad (158)$$

Suppose that, in addition to permutation invariance of the n uses, each *use* of the channel is G -covariant in the sense that

$$\mathcal{E}_{B^n|A^n}g_{A_j|A_j} = g_{B_j|B_j}\mathcal{E}_{B^n|A^n} \quad (159)$$

for all $g \in G$ and $j \in \{1, \dots, n\}$. Here the representations of G on each system are the same, except that they act on different systems. This is the case, for example, if $\mathcal{E}_{B^n|A^n}$ is n uses of a G -covariant memoryless channel.

To every ordered pair (π, \mathbf{g}) where $\pi \in S_n$ and $\mathbf{g} \in G^{\times n}$, we can associate an action $(\pi, \mathbf{g})_{A^n|A^n} := \pi_{A^n|A^n}\mathbf{g}_{A^n|A^n}$. Here the action on A^n for $\mathbf{g} = (g^{(1)}, \dots, g^{(n)})$ is $\mathbf{g}_{A^n|A^n} := \bigotimes_{j=1}^n g_{A_j|A_j}^{(j)}$ and $\mathbf{g}_{B^n|B^n} := \bigotimes_{j=1}^n g_{B_j|B_j}^{(j)}$. Under composition these actions constitute a group, which is a semi-direct product of $G^{\times n}$ and S_n ($G^{\times n}$ being the normal subgroup) which we denote $S_n \ltimes G^{\times n}$. Defining the action of $(\pi, \mathbf{g}) \in S_n \ltimes G^{\times n}$ on states of B^n by $(\pi, \mathbf{g})_{B^n|B^n} := \pi_{B^n|B^n}\mathbf{g}_{B^n|B^n}$, we have

$$\mathcal{E}_{B^n|A^n}(\pi, \mathbf{g})_{A^n|A^n} = (\pi, \mathbf{g})_{B^n|B^n}\mathcal{E}_{B^n|A^n}. \quad (160)$$

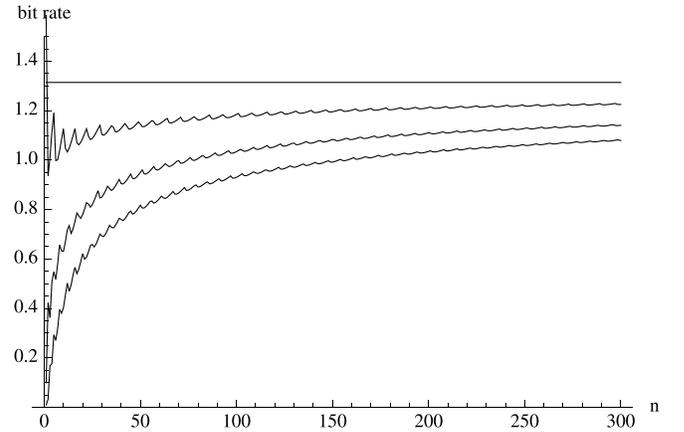


Fig. 4. Our upper bound (Theorem 18) on the rate of entanglement-assisted codes evaluated (see Section VI) for three different error probabilities ϵ , for the qubit depolarising channel with failure probability 0.15. From bottom up, the curves are for the error probabilities 0.001, 0.01 and 0.1. The top-most line marks the capacity of the channel (roughly 1.31 bits/channel use) as given by the formula of Bennett, Shor, Smolin and Thapliyal [12].

VI. EXAMPLE: THE DEPOLARISING CHANNEL

A single use of the d -dimensional depolarising channel with parameter p and d -dimensional input and output systems A and B has the operation

$$\mathcal{D}_{B|A}\tau_A = (1-p)\text{id}_{B|A}\tau_A + p\mu_B\text{Tr}_A\tau_A, \quad (161)$$

where, for any system Q , $\mu_Q := \mathbb{1}_Q/\dim(Q)$ denotes the maximally mixed state on that system. For n uses the operation is

$$\mathcal{D}_{B^n|A^n}^{\otimes n} = \mathcal{D}_{B_1|A_1} \dots \mathcal{D}_{B_n|A_n}, \quad (162)$$

which has the covariance group $S_n \ltimes U(d)^{\times n}$.

The only input and output states with the corresponding invariances are the maximally mixed states. Therefore,

$$\sup_{\rho_{A^n}} I_\epsilon^{\text{ALL}}(\mathcal{D}_{B^n|A^n}^{\otimes n}, \rho_{A^n}) = D_\epsilon^{\text{ALL}}(\phi(p)_{\tilde{A}^n B^n}^{\otimes n} \|\mu_{\tilde{A}^n}\mu_{B^n}) \quad (163)$$

where $\phi(p)_{\tilde{A}B} := \mathcal{D}_{B|A}\Phi_{\tilde{A}\tilde{A}}/d = (1-p)\Phi_{\tilde{A}B}/d + p\mu_{\tilde{A}}\mu_B$ is an *isotropic* state. Since the arguments of D_ϵ^{ALL} commute in this expression, this is equivalent to a classical hypothesis test between the distributions given by the eigenvalues of the two states. In fact, the degeneracy of the eigenvalues makes it equivalent to deciding between hypotheses on the distribution of n samples of a binary variable: Hypothesis H_0 is that the samples are drawn i.i.d. with probability $(1-p) + p/d^2$ of being 0, and hypothesis H_1 is that the samples are drawn i.i.d. with probability $1/d^2$ of being 0. Therefore,

$$\begin{aligned} D_\epsilon^{\text{ALL}}(\phi(p)_{\tilde{A}^n B^n}^{\otimes n} \|\mu_{\tilde{A}^n}\mu_{B^n}) \\ = D_\epsilon^{\text{ALL}}((\mu, 1-\mu)^{\otimes n}, (\lambda, 1-\lambda)^{\otimes n}) \end{aligned} \quad (164)$$

where $\mu = (1-p) + p/d^2$ and $\lambda = 1/d^2$, and Proposition 32 gives a formula for this quantity which is easy to evaluate exactly (as we have done for Fig. 4).

Proposition 32: Let $\mu \geq \lambda$ be two probabilities.

$$\beta_\epsilon((\mu, 1 - \mu)^{\otimes n}, (\lambda, 1 - \lambda)^{\otimes n}) = (1 - \gamma)\beta_{\ell(\epsilon)} + \gamma\beta_{\ell(\epsilon)+1} \quad (165)$$

where

$$\alpha_\ell = \sum_{j=0}^{\ell-1} \binom{n}{j} \mu^j (1 - \mu)^{n-j}, \quad (166)$$

$$\beta_\ell = \sum_{j=\ell}^n \binom{n}{j} \lambda^j (1 - \lambda)^{n-j}, \quad (167)$$

$\ell(\epsilon)$ is the value of l satisfying $\alpha_\ell \leq \epsilon \leq \alpha_{\ell+1}$ and $\gamma = (\alpha - \alpha_{\ell(a)})/(\alpha_{\ell(a)+1} - \alpha_{\ell(a)})$.

Proof: This is just optimising over the optimal (classical) hypothesis tests identified by the Neyman-Pearson lemma. The same expression is given in [6]. ■

VII. CONVERSES AND CRYPTOGRAPHY

In [23] a strong converse was proven for the classical capacity of many quantum channels \mathcal{N} , including depolarising noise. The results of [23] had a nice application to proving security in so-called noisy-storage model of quantum cryptography [24]. This model allows for the secure implementation of any two-party cryptographic task under the assumption that the adversary's quantum memory is noisy. Examples of such tasks include bit commitment and oblivious transfer, which are impossible to achieve without assumptions [25], [26].

Concretely, the noisy-storage model assumes that during waiting times introduced into the protocol, the adversary can only store quantum information in a memory device modelled by a channel $\mathcal{E}_{B|A}$. Otherwise the adversary is all powerful and may even use a quantum computer to perform the most advantageous error-correcting encoding. At the beginning of the waiting time, the state of the protocol can be described as a cq-state ρ_{XKA} where X is a classical register held by the honest party and storing a string X and A and K are the quantum and classical registers of the dishonest party respectively. After the wait time, the state of the protocol is described by $\mathcal{E}_{B|A}\rho_{XKA}$. The security of all protocols proposed in this model requires a bound on the min-entropy $H_{\min}(X|BK) = -\log P_{\text{guess}}(X|BK)$, where $P_{\text{guess}}(X|BK)$ is the probability that the adversary holding B and K manages to guess X maximized over all possible measurements on BK . Bounds on this quantity can be linked to the classical capacity [24], the entanglement cost [27] or the quantum capacity [28] of $\mathcal{E}_{B|A}$. Yet, explicit bounds on the min-entropy which would enable practical implementations of such protocols [29], [30] are elusive. In particular, up to this date no statements are known for arbitrary channels $\mathcal{E}_{B|A}$, such as channels which do not obey a strong converse or simply structureless channels. Our method can be used to compute explicit bounds on the min-entropy for arbitrary channels.

The key to relating our analysis to a study of the min-entropy is the observation [24] that

$$H_{\min}(X|BK) \quad (168)$$

$$\geq -\log \max_{\mathcal{Z}} \Pr \left(\hat{W} = W | \mathcal{E}_{B|A}, \mathcal{Z}, S_{[H_{\min}(X|K)]} \right). \quad (169)$$

That is, by understanding the adversary's knowledge $H_{\min}(X|K)$ conditioned on the classical knowledge K alone, and the properties of the channel $\mathcal{E}_{B|A}$ we can bound the adversary's knowledge about a string X given both B and K . In [24] the bound $H_{\min}(X|K) \gtrsim n/2$ was obtained from an uncertainty relation for BB84 measurements that was used to generate the n -bit string X . Different measurements lead to higher (or lower) values of

$$\frac{H_{\min}(X|K)}{n} =: \hat{R}. \quad (170)$$

In [24], a strong converse for some channels $\mathcal{N}^{\otimes \ell}$ was then used to bound the r.h.s. for $\mathcal{E}_{B|A}$ and rate $R = (n\hat{R})/\ell$, as long as R exceeded the capacity of \mathcal{N} .

Let us now sketch how our approach directly leads to a security statement for any $\mathcal{E}_{B|A}$. More specifically, we will turn things around, fix \hat{R} , and ask how large we have to choose n such that sending $n\hat{R}$ randomly chosen classical bits through the channel $\mathcal{E}_{B|A}$ will incur an error of at least ϵ . Intuitively, our goal is to effectively overflow the adversary's storage device $\mathcal{E}_{B|A}$, that is, n will be chosen such that no coding scheme allows for an error less than ϵ for a fixed value of \hat{R} . By the results of [24] we can then bound the adversary's min-entropy as $H_{\min}(X|BK) \geq -\log(1 - \epsilon)$. For any $\epsilon > 0$, our analysis yields a bound on $\log M_\epsilon$ stating that if we were to transmit more than $n\hat{R} > \log M_\epsilon$ bits, the error is necessarily $\epsilon + \delta > \epsilon$ for some $\delta > 0$, which yields the desired bound. As \hat{R} is determined by an uncertainty relation, we thus know how many transmissions n we have to make to obtain security. Note that allowing the adversary to perform entanglement-assisted coding only gives him additional power, and hence our analysis for entanglement-assisted coding provides a method with which explicit security parameters may be computed by means of a semi-definite program.

VIII. CONCLUSION

We have shown how a simple and powerful idea [6] for obtaining a finite blocklength converse for classical channels in terms of a hypothesis testing problem can be generalised to quantum channels and entanglement-assisted codes.

This generalisation has the property that a natural restriction on codes (removing entanglement-assistance) translates into a natural restriction on the tests that can be performed in the hypothesis testing problem (they must be local). This provides a strong link between the extensively studied problems of channel coding and hypothesis testing (and state discrimination) of bipartite systems under locality restrictions.

Many avenues for further work are apparent to the authors: Subsection V-E invites a more thorough investigation into the extent to which symmetries can be used to simplify evaluation of the bound in various special cases, and especially in the case of general memoryless channels, where one might hope for an exponential reduction in the size of the SDP, in a quantum generalisation of [13].

While we have been able to fit the existing converse of Wang and Renner [2] precisely into our hierarchy bounds based on restricted hypothesis testing, it is not obvious to the authors what relationship exists between our converse for

entanglement-assisted codes and that of Datta and Hsieh [4]. We would like to know what can be said about this, particularly in light of the achievability bound given in [4].

A limitation of our work is that we only generalise the classical bound of [6] for the case of finite input/output alphabets (as our input/output systems have finite dimension). To analyse the general case will require greater mathematical sophistication (see [7]), but would be desirable given (for example) the interest in quantum gaussian channels [31].

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REFERENCES

- [1] M. Mosonyi and N. Datta, "Generalized relative entropies and the capacity of classical-quantum channels," *J. Math. Phys.*, vol. 50, no. 7, p. 072104, 2009.
- [2] L. Wang and R. Renner, "One-shot classical-quantum capacity and hypothesis testing," *Phys. Rev. Lett.*, vol. 108, no. 20, p. 200501, May 2012.
- [3] J. M. Renes and R. Renner, "Noisy channel coding via privacy amplification and information reconciliation," *IEEE Trans. Inf. Theory*, vol. 57, no. 11, pp. 7377–7385, Nov. 2011.
- [4] N. Datta and M.-H. Hsieh. (May 2011). "One-shot entanglement-assisted quantum and classical communication." [Online]. Available: <http://arxiv.org/abs/1105.3321>
- [5] N. Datta, M. Mosonyi, M.-H. Hsieh, and F. G. S. L. Brandao, "Strong converse capacities of quantum channels for classical information," *IEEE Trans. Inf. Theory*, vol. 59, p. 8014, 2013, arXiv:1106.3089v1.1109/TIT.2013.2282160.
- [6] Y. Polyanskiy, H. V. Poor, and S. Verdú, "Channel coding rate in the finite blocklength regime," *IEEE Trans. Inf. Theory*, vol. 58, no. 5, pp. 2307–2359, May 2010.
- [7] Y. Polyanskiy, "Saddle point in the minimax converse for channel coding," *IEEE Trans. Inf. Theory*, vol. 59, no. 5, pp. 2576–2595, May 2013.
- [8] E. M. Rains, "Rigorous treatment of distillable entanglement," *Phys. Rev. A*, vol. 60, no. 1, pp. 173–178, Jul. 1999.
- [9] E. M. Rains, "A semidefinite program for distillable entanglement," *IEEE Trans. Inf. Theory*, vol. 47, no. 7, pp. 2921–2933, Nov. 2001.
- [10] S. Virmani and M. B. Plenio, "Construction of extremal local positive-operator-valued measures under symmetry," *Phys. Rev. A*, vol. 67, no. 6, p. 062308, Jun. 2003.
- [11] G. M. D'Ariano, P. L. Presti, and P. Perinotti, "Classical randomness in quantum measurements," *J. Phys. A, Math. General*, vol. 38, no. 26, p. 5979, 2005.
- [12] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal, "Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem," *IEEE Trans. Inf. Theory*, vol. 48, no. 10, pp. 2637–2655, Oct. 2002.
- [13] W. Matthews, "A linear program for the finite block length converse of Polyanskiy–Poor–Verdú via nonsignaling codes," *IEEE Trans. Inf. Theory*, vol. 58, no. 12, pp. 7036–7044, Dec. 2012.
- [14] Y. Polyanskiy and S. Verdú, "Arimoto channel coding converse and Rényi divergence," in *Proc. 48th Annu. Allerton Conf. Commun., Control, Comput. (Allerton)*, Sep./Oct. 2010, pp. 1327–1333.
- [15] N. Sharma and N. A. Warsi, "Fundamental bound on the reliability of quantum information transmission," *Phys. Rev. Lett.*, vol. 110, no. 8, p. 080501, Feb 2013.
- [16] M. Hayashi, *Quantum Information: An Information*. New York, NY, USA: Springer-Verlag, 2006.
- [17] L. Vandenberghe and S. Boyd, "Semidefinite programming," *SIAM Rev.*, vol. 38, no. 1, pp. 49–95, 1996.
- [18] A. S. Holevo, "Bounds for the quantity of information transmitted by a quantum communication channel," *Problemy Peredachi Inf.*, vol. 9, no. 3, pp. 3–11, 1973.
- [19] B. Schumacher and M. D. Westmoreland, "Optimal signal ensembles," *Phys. Rev. A*, vol. 63, no. 2, p. 022308, Jan. 2001.
- [20] A. S. Holevo, "The capacity of the quantum channel with general signal states," *IEEE Trans. Inf. Theory*, vol. 44, no. 1, pp. 269–273, Jan. 1998.
- [21] B. Schumacher and M. D. Westmoreland, "Sending classical information via noisy quantum channels," *Phys. Rev. A*, vol. 56, no. 1, pp. 131–138, Jul. 1997.
- [22] C. Adami and N. J. Cerf, "von Neumann capacity of noisy quantum channels," *Phys. Rev. A*, vol. 56, no. 5, pp. 3470–3483, Nov. 1997.
- [23] R. König and S. Wehner, "A strong converse for classical channel coding using entangled inputs," *Phys. Rev. Lett.*, vol. 103, no. 7, p. 070504, Aug. 2009.
- [24] R. König, S. Wehner, and J. Wullschleger, "Unconditional security from noisy quantum storage," *IEEE Trans. Inf. Theory*, vol. 58, no. 3, pp. 1962–1984, Mar. 2012.
- [25] H.-K. Lo and H. F. Chau, "Why quantum bit commitment and ideal quantum coin tossing are impossible," *Phys. D, Nonlinear Phenomena*, vol. 120, nos. 1–2, pp. 177–187, 1998.
- [26] D. Mayers, "Unconditionally secure quantum bit commitment is impossible," *Phys. Rev. Lett.*, vol. 78, no. 17, pp. 3414–3417, Apr. 1997.
- [27] M. Berta, M. Christandl, F. G. S. L. Brandao, and S. Wehner, "Entanglement cost of quantum channels," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jul. 2012, pp. 900–904.
- [28] M. Berta, O. Fawzi, and S. Wehner, "Quantum to classical randomness extractors," in *Advances in Cryptology (Lecture Notes in Computer Science)*, vol. 7417, R. Safavi-Naini and R. Canetti, Eds. Berlin, Germany: Springer-Verlag, 2012, pp. 776–793.
- [29] N. H. Y. Ng, S. K. Joshi, C. C. Ming, C. Kurtsiefer, and S. Wehner, "Experimental implementation of bit commitment in the noisy-storage model," *Nature Commun.*, vol. 3, Nov. 2012, Art. ID 1326.
- [30] C. Erven, N. Ng, N. Giggov, R. Laflamme, S. Wehner, and G. Weihs, "An experimental implementation of oblivious transfer in the noisy storage model," *Nature Commun.*, vol. 5, Feb. 2014, Art. ID 3418.
- [31] A. S. Holevo, M. Sohma, and O. Hirota, "Capacity of quantum Gaussian channels," *Phys. Rev. A*, vol. 59, no. 3, pp. 1820–1828, Mar. 1999.

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